# THE EVOLUTION OF NUMBER IN MATHEMATICS 

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#### Abstract

The number concept has evolved in a gradual process over ten millenia from concrete symbol association in the Middle East c. 8000 BCE (" 4 sheep-tokens are different than 4 grain-tokens") ${ }^{1}$ to an hierarchy of number sets $(\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O})^{2}$ the last four of which unify number with space and motion. ${ }^{3}$ In this paper we trace this evolution from whole numbers at the dawn of accounting with number-tokens through the paradoxes of algebraic irrationals, non-solvability by radicals ${ }^{4}$, and uncountably many transcendentals, to the physical reality of imaginary numbers and the connection of the octonions with supersymmetry and string theory. ${ }^{5}$ (FAQ3) Our story is of conceptual abstraction, algebraic extension, and analytic completeness, and reflects the modern transformation of mathematics itself. The main paper is brief (four pages) followed by supplements containing FAQs, historical notes, and appendices (holding theorems and proofs), and an annotated bibliography. A project supplement is in preparation to allow learning the material through self-discovery and guided exploration.


For scholars and laymen alike it is not philosophy but active experience in mathematics itself
that can alone answer the question: 'What is Mathematics?' Richard Courant (1941), book of the same title
"One of the disappointments experienced by most mathematics students is that they never get a [unified] course on mathematics. They get courses in calculus, algebra, topology, and so on, but the division of labor in teaching seems to prevent these different topics from being combined into a whole. In fact, some of the most important and natural questions are stifled because they fall on the wrong side of topic boundary lines. Algebraists do not discuss the fundamental theory of algebra because "that's analysis", and analysts do not discuss Riemann surfaces because "that's topology," for example. Thus, if students are to feel they really know mathematics by the time they graduate, there is a need to unify the subject." John Stillwell (1989), Mathematics and Its History

Over the past ten millenia, a gradual process of conceptual abstraction and algebraic and analytical extension has transformed mathematics and unified counting ${ }^{6}$, motion, and space into a generalized concept of number. Number here refers to quantities which can be combined through computation using one or more operations.(FAQ X2) ${ }^{7}$

The whole numbers have long been familiar to all who can count. But though the archaeological evidence for counting dates to at least 8,000 B.C.E,* recent research on the Piraha tribe of the Brazilian Amazon (2004) [24] ${ }^{11} 121314$ has shown that the development of counting and its linguistic expression are not universal amongst all human cultures, contrary to previous belief. ${ }^{15}$ For those cultures that adopt numeracy, ${ }^{16}$ the arithmetic operations, negative numbers, and rational fractions become familiar through accounting transactions (commerce and banking) ${ }^{17}$, statal administration,(apportionment, taxation, redistribution) ${ }^{18}$ metrology (measurement and standards), surveying (geometrical measures and curvature), and the creation of a viable calendar. ${ }^{\dagger}$ These experiences are reflected, albeit in an impoverished form, in the arithmetic and algebra taught in schools. ${ }^{2021}$ But amongst them lie seeds for further mathematical discovery. ${ }^{22}$ Let us look more closely.

The rational numbers $\mathbb{Q}$ are in theory sufficient for all of engineering and applied science, since every measured number has finite precision and therefore must itself be rational. Indeed, we have that a rational number has either a finite expansion, or if infinite, has a finite block that repeats indefinitely.(Appendix C) ${ }^{23}$ But the rationals are an infinite set (FAQ5), so no finite precision computing machine could represent them all. ${ }^{24}$ As such, computational mathematics uses the large but finite set of floating point numbers.

[^0]These are able to represent both very large and very small numbers with a known maximum error for numbers falling within range. ${ }^{25}$

The pressure to extend the exact numbers (those having no error) beyond the rationals comes from 1) numerical mathematics via error analysis, 2) geometry via incommensurability of length and area, 3) algebra via the desire for algebraic closure and for solving algebraic polynomials, and 4) analysis via completeness (freedom from 'holes') and connectedness (continuous paths connecting any two points) required to model the continuum.

These constraints will take us as far as the one-dimensional reals $\mathbb{R}$ and two-dimensional complex numbers $\mathbb{C}$. But beyond this, it is curiosity and the intriguing possibility of a higher dimensional analogue to $\mathbb{C}$ that culminates in the four-dimensional quaternions $\mathbb{H}$ and eight-dimensional $\mathbb{O}$. But as often in mathematics, the results of a pure thought experiment find an application. The octonions, discovered in 1843, have earned a place in 20th century applied maths by explaining why this tower of algebras progresses in powers of 2 , why it stops at 8 dimensions (FAQ3), and now, in the 21st century the possibility that the 8 -dimensional number system may be the best language for describing the fundamental 'grand unified theories' of the universe. But first things first.

Computationally, exact numbers are required to improve floating point numerical algorithms and identify the best arrangement of calculations to control the error that accumulates during extended computations. ${ }^{26}$

But geometry shows that the rational exact numbers are not plentiful enough to include many quantities which are undeniably qualified to be considered as 'number', despite there being infinitely many rationals, both at the large and small scales, and the property of Archimedes which says between any two rationals there is always another. In other words, there is no smallest quantum of granularity between rational numbers.

The existence of demonstrably irrational numbers should come as a shock-as it was for the Greeksbut once one is found, then there are a whole lot more: these appear commonly as geometric lengths (e.g. diagonals of squares and cubes e.g. $\sqrt{2}$ and $\sqrt[3]{2}$ ), ratios of lengths (e.g. circumference to diameter $\pi,{ }^{27}$ and the golden ratio*), chords of circles ( $2 \sin \theta$ for rational $\theta$, i.e. not multiples of $\pi$ ), arclengths of ellipses, ${ }^{28}$ rates (e.g. growth rate $e$ under continuous compounding), or as binary decimal expansions whose digits encode a parameterized decision problem (e.g. setting the nth binary digit to 1 if the nth integer is prime, else 0 ).

Some of these are algebraic numbers, i.e. roots of polynomials with rational coefficients (integers if denominators are cleared), but many are not, i.e. are transcendental. Yet all of these irrational quantities exist in the sense that they can be defined precisely and computed to arbitrary precision using rational numbers (typically using iteration) even though they themselves are demonstrably not rational.(FAQ9)

While the rationals are closed with respect to arithmetic (plus, minus, multiply, divide, power) they are not closed with respect to algebraic operations (root). For example, it is not possible to solve every algebraic equation having rational coefficients within the rationals. Consider that $x^{2}-c=0$ has no rational solutions either when $c<0$ or $c$ is prime, since $\mathbb{Q}$ contains neither $\sqrt{-1}=i$ nor $\sqrt{p}$ for any prime $p .^{29}$

The simplest expansion is by field extension $\mathbb{Q}[F]$, where $F$ is the set of constants required to keep the system closed, e.g. $\mathbb{Q}[i:=\sqrt{-1}]$ or $\mathbb{Q}\left[\left\{\sqrt[3]{2}, \sqrt[3]{2}^{2}\right\}\right]$. Expanding the rational numbers $\mathbb{Q}$ to include these 'new' numbers requires defining what addition and multiplication look like so as to 1 ) preserve the relations of the existing numbers, and 2) ensure that what's added preserves arithmetic closure. ${ }^{30}$ Multiplication within the algebraically extended system is defined by treating each number as a binomial in the algebraic constants $x \in F$.

Listing the constructions adding numbers to $\mathbb{Q}$, we have:
(1) finite field extensions, e.g. by $\sqrt{2},\left\{\sqrt[3]{2}, \sqrt[3]{2}^{2}\right\}$, $i$, or indeed a fixed literal $x$;
(2) constructible numbers: intersections obtained through a finite number of operations with a straight-edge and compass, e.g. $\sqrt{2}$;
(3) algebraic numbers: all solutions of polynomials with rational coefficients-note this includes $i=\sqrt{-1}, \sqrt[3]{2}$ $\left(x^{3}-2=0\right)$;
(4) periodic numbers: integrals of algebraic functions which includes non-algebraic transcendental numbers such as arc-length of an ellipse (elliptic integrals 38, 40]);
(5) computable numbers: for which a finite terminating algorithm can be given for calculating the number to arbitrary precision, e.g. $\pi, e^{r}=\lim _{n \rightarrow i n f t y}(1+r / n)^{n}, \sqrt{\pi}, e^{\pi} ;{ }^{31}$ and
(6) definable numbers: any numbers which can be defined using first order logic (arithemtically definable) or second or higher order logic (analytically definable); ${ }^{32}$
At this stage we have gone as far as we can go with a constructivist understanding of number. We have a number system that includes every known mathematical constant, all named transcendental numbers ${ }^{33}$,

[^1]and indeed even every potentially definable number. But while significantly expanded, our number set is still countably infinite, i.e. can be placed in one-to-one correspondence with the whole numbers. ${ }^{34}$

Countable infinity - the cardinality of the rationals and the only form of infinity that the constructivist approach allows - turns out to be a limiting condition to the birth of the continuum. We will be forced to conclude that there is no way to obtain the continuum without triggering the admittance of the full vastness of the so-called 'uncountable infinity'. ${ }^{35}$

Analytically, we want a suitable model for the geometric continuum, i.e. we want a guarantee that our number system does not have any holes or gaps. Connectedness is the requirement that there is a continuous path between every pair of points $(a, b)$ in a set $S$, such that the unit interval $[0,1]$ can be mapped continuously into $S$. In symbols: $\exists f(t):[0,1] \rightarrow S$, s.t. $f(0)=a, f(1)=b .{ }^{36}$

It turns out that obtaining an analytically complete concept of number forces upon us a new set which is enormous beyond imagining. So long as we do not allow a somehow larger notion of infinity, there are simply not enough points with which to create a continuum, despite the expansions listed above to the rationals (FAQ5). This will mark the first encounter with a higher order of infinity, the uncountable infinity (FAQ5).

Our route to constructing the continuum (following Cantor) is to explicitly define, as a distinct 'number, every convergent infinite sequence of rationals whose limit is distinct (alternatively Dedekind cut), then to define the arithmetic combination of these, and finally to show that their totality is arithmetically closed and therefore by definition analytically complete. ${ }^{37}$ The resulting set of 'numbers' (actually convergent sequences) is denoted by $\mathbb{R}$ and called the 'real' numbers, as it is unique under isomorphism. With this is constructed a precise mathematical model for both space (the continuum) and time (infinitely divisible durations). ${ }^{38}$

Whereas the first shock was that the rationals are insufficient for geometry, there is now a second shock: Cantor's diagonal argument (FAQ5) shows that $\mathbb{R}$ cannot be put into one-to-one correspondence with the rationals, meaning that it is of a so-called uncountable infinity, i.e. a higher order of infinity than the countable infinity. This is because $\mathbb{R}$ was built to include the set of all possible decimal expansions in between every unit interval, i.e. all possible infinite sequences of digits. and so contains essentially the powerset of the naturals. And we know that the powerset of a set, whether finite or infinite, can never be put into one-to-one correspondence with its generating set, i.e. in symbols, $|\wp(S)| \neq|S| \forall S$.

We are thus forced to accept the disconcerting fact that by filling in all possible gaps to ensure the continuum (spatially continuous model), we have introduced a vast, uncountable infinity of numbers which can neither be computed nor even defined.

An example of the uncountably infinite new numbers that have been added into $\mathbb{R}$, but which cannot be constructed, computed, or even defined, is Chaitin's constant, whose binary encoding is based on deciding a sequence of halting problems. Since the halting problem is itself undecidable, the number is therefore also undefinable. ${ }^{39}$

This vastness is a source of many paradoxes ('monsters') in analysis and topology. ${ }^{40}$ As an example, the open interval $(-\epsilon,+\epsilon) \forall \epsilon>0$ can be shown to have the same cardinality as the entirity of $\mathbb{R}$ and furthermore the same cardinality as $\mathbb{R}^{n} \forall n \in \mathbb{N}$ (this is shown using a space- and volume- filling construction due to Peano).

Cardinality, not dimension, is now the key concept that determines the true size of sets. The result is a collapse of infinite sizes into a countably infinite hierarchy, with countable infinity as the smallest infinite cardinality $\aleph_{0}$, and each power-set of the previous infinite cardinal gives the next, i.e. $\aleph_{1}=2^{\aleph_{0}}$, $\aleph_{2}=2^{\aleph_{1}}, \ldots$

An unresolved question is whether there exists a set and cardinality greater than countable $\aleph_{0}$ but smaller than the cardinality of the continuum $\mathfrak{c}=2^{\aleph_{0}}=\aleph_{1}$, i.e. whether the next largest infinite cardinal is that of the continuum. This is the Continuum Hypothesis, shown to be independent of the current standard model of set theory axiomatized by the ZFC axioms. One view among logicians is that settling CH one way or the other will require additional axioms for set theory - but what their justification might be is not yet clear. ${ }^{41}$

To bring algebraic closure to $\mathbb{R}$, we must add the complex field extension with the usual definition of addition of like symbols, and the binomial definition of multiplication described earlier. This forms the set of complex numbers $\mathbb{C}=\mathbb{R}[i:=\sqrt{-1}]$, also definable as a pair $(a, b)$ of real numbers in a vector space with basis $\{1, i\}$.
$\mathbb{C}$ is the smallest analytically complete and algebraically closed field. All complex algebraic operations are now permitted, including taking roots of negative real numbers, and powers and roots of rationals, reals, and even complex numbers themselves. Importantly, every polynomial now splits into linear factors in $\mathbb{C}$ or, in other words, has all its roots (the Fundamental Theorem of Algebra). But algebraic closure for $\mathbb{C}$ comes at the cost of losing well-ordering and increasing the occurrence of branch cuts. ${ }^{42}$
$\mathbb{C}$ has another remarkable property when viewed geometrically: multiplying by a complex number $z$ is equivalent to inducing motion: a stretching when $z$ is pure real, rotation when pure imaginary, and both together when $z$ is general. ${ }^{43}$ This leads to a direct analogy between complex numbers and two-by-two matrices:

$$
z=(x+i y)=r e^{i \theta}=\left(\begin{array}{cc}
x & -y \\
y & x
\end{array}\right)=r\left(\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right)
$$

Inspired by this success with $\mathbb{C}$, we discover the quaternions $\mathbb{H}$ as a pair of complex numbers to be an algebraically closed field of dimension four, interpretable as stretching and rotation in three-dimensional space, and reinforcing the link between number and motion. The geometric link means $\mathbb{H}$ can also be modelled using $3 \times 3$ matrices. But this is gained at a cost of losing commutativity.

Going further still, we discover the octonions $\mathbb{O}$ of dimension eight as a pair of quaternions, where now associativity is sacrificed in their multiplication. ${ }^{44}$ Remarkably, this construction in powers of 2 stops here - © are the highest dimension possible normed division algebra. Equally remarkably, octonions have deep connections to physics string theory and geometry. Further generalizations of the number concept are vectors, matrices, tensors, and abstract rings ${ }^{45}$ and groups. (FAQ10) ${ }^{46}$

We have seen that the concept of number in modern mathematics unifies the discrete notion of counting with the continuous and dynamic notions of motion in space, plus the algebraic structure of a division algebra.

Space can also be described using the continuum model as a collection of continuous coordinates using tuples of numbers forming multi-dimensional vector spaces, with coordinates chosen from $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$. Time is described via infinitely divisible real quantities. Motion is perhaps the most striking feature about the modern notion of number as transformation: each of these number towers is essentially rotation and stretching in 2 -, 3 - and 8 -dimensional space. ${ }^{47}$

The process of going from the whole numbers to the octonions has traced through almost every major achievement in mathematics over the past five millenia. To understand deeply the generalized concept of number, in all of its facets, is to understand a good portion of mathematics.

Epilogue 1. It is noteworthy that against this millenia-long experience with number, it is only in the past 100 years, since Emmy Noether's 1921 paper on "The Ideal Theory in Rings" ${ }^{48}$, that the essential properties of our number collections have been extracted into an abstract algebraic setting that allows the commonalities of computation amongst a host of concrete examples to be recognized as fundamentally alike, whether the objects of computation are numerical or not. The positive whole numbers $\mathbb{N}^{+49}$ generalize to semi-rings, the integers $\mathbb{Z}^{50}$ to monoids ${ }^{51}$, the rationals $\mathbb{Q}^{52}$ to rings, the reals $\mathbb{R}$ and complexes $\mathbb{C}$ (which we shall encounter again below) to fields, the quaternions $\mathbb{H}$ to division rings ${ }^{53}$, and the octonions $\mathbb{O}$ to nonassociative division algebras. As such, the computational behavior of wildy different mathematical objects such as matrices, polynomials, permutations, series, rotations, functions, are entirely determined by the axioms of the class within which they fall.(FAQ4) An extreme example of a non-numerical mathematical object that follows numerical computation rules is Rock-paper-scissors which is a finite magma, is closed, commutative, non-associative, has neither inverses nor identity elements, and yet computations within this structure follow clear rules known by school children across the Far East, Europe, and USA. ${ }^{54}$

Epilogue 2. If number stayed with counting, and arithmetic (division), measurement along a straight line, and only linear operations, one would have been able to stay with rational numbers. And yet even here there is deep mathematics: primality, the distribution of the primes, and solution of diophantine equations, i.e. solution of equations in integers (though this is solvable for linear diophantine equations through modular arithmetic).

It is the moment one allows space to expand into a second dimension to get squares and circles and second order operations (i.e. power, for area and for Euclidean distance via Pythagorean theorem), then one is forced to admit the irrationals - diagonal of square, and circumference of a unit circle, circumferance of an ellipse. This forces recognition that the rationals are defective for modelling the geometric continuum, which begins the movement to the reals, and the defectiveness of the reals for solving higher order equations, which introduces the complex numbers with their interpretation as rotation. One has further deep mathematics: the fundamental theorem of algebra on the one hand, and yet the insolubility by rationals of the general polynomial (Abel's theorem)

One might reasonably ask, what are the physical motivations for powers higher than 4, and indeed in direct physical models, these are hard to find ${ }^{55}$. But then one observes that the collection of polynomials $x^{n}$ forms a basis for the space of functions, and therefore all functions are now able to be approximated by polynomials, much like the reals can be approximated by the rationals. It turns out that the polynomials are dense in the space of all continuous and infinitely differentiable (smooth) functions. So in this sense
then polynomials serve as the building block of functions analogous to the rationals as the building block of number.

Once one has gotten to a geometry of 2 dimensions with the field of complex numbers it is the desire for dimensional extension - can one get to higher dimensional fields. And again remarkable mathematics: no such structures except in dimensions that are powers of 2 , and then only up to the octonions. The sedenions get zero divisors.

For the rest of space, one needs then coordinates, tuples, vectors, and matrices, which then also generalize rotations, reflections, stretching, into higher dimension. The equivalent to numbers among the matrices are the Lie groups, or $\mathrm{SO}(\mathrm{n})$ among the reals and $\mathrm{SU}(\mathrm{n})$ for the complexes. These are all invertible, etc. and form a field. So now we have the connection between matrices and number.

## Notes

${ }^{1}$ the cardinality of a finite set of specific objects
${ }^{2}$ and an uncountable hierarchy of infinite cardinalities $\aleph_{0}, \aleph_{1}, \aleph_{2}, \ldots$. In this paper we are less concerned with the infinite cardinals, though these form a limited arithmetic.
${ }^{3}$ Compared to their origins, the final concept of number is generalized and unifies arithmetic, algebra, geometry and physics. Space is present in the concept of the continuum and coordinates $\left(\mathbb{R}^{n}\right)$, motion in Lie groups of rotations and stretchings. Both scales are also represented: the infinitesimally small already in the rationals, and a hierarchy of infinite cardinalities that extends into the unimaginably large, with ever higher orders of infinity, each drawn from the power set of the preceding cardinality.
${ }^{4}$ Some solutions to polynomials of degree five or greater cannot be expressed as radicals
${ }^{5}$ The 1843 discoveries of quaternions (Hamilton) and octonions (Graves and Cayley), and the connection of octonions to quantum mechanics (1934, von Neumann and Wigner), to supersymmetry (1983, Kugo), and to string theory (1987, Green, Schwarz, and Witten).
${ }^{6}$ Measurement is fundamentally counting because it requires as reference a fundamental unit of measure.
${ }^{7}$ These operations can be binary, such as + or $\times$ or unary such as $x^{-1}$. Quantities are not required to have inverses, for example in rock-paper-scissors, though in general, we typically expand systems so that every operation can be reversed, since in most cases we want to have a way back: if we go from A to B, we typically also want a way back to A from B. Unique inverses have their advantages, but we are willing to accept that there could be multiple ways back for example different paths or zero-divisors, i.e. two non-zero matrices multiplied to get zero matrix (exercise).
${ }^{8}$ The Lebombo bone, a baboon fibula discovered in a cave in the mountains between South Africa and Swaziland dating back to 35,000 B.C.E, holds claim to be the oldest mathematical artifact and is believed by some to have been a tally stick or calendar stick due to 29 notches corresponding to the lunar or female fertility cycle.(Bogoshi, Mathematical Gazette, 1987).
${ }^{9}$ The Ishango bone, a $10-\mathrm{cm}$ bone that used to be a tool handle, was discovered near Lake Edward between Uganda and Zaire, and is believed to be hold the earliest known collection of small primes.


Figure 1. Drawings of the markings on the Ishango bone and their conjectured relationship to primes.


Figure 2. Photographs of the Ishango bone.

[^2]${ }^{11}$ The anthropological findings [15] documenting the absence of counting and indeed many other unique aspects of the cultural constraints of the Piraha, come from Daniel Everett, a linguist who lived immersed with the Piraha for 6 years 7 (since 1977) becoming completely familiar with their language and cultural practice, and with whom he has maintained regular contact every year for over 20 years. The Piraha, a population of less than 200, live on the banks of the Maici River in the Lowland Amazonia region of Brazil, in small villages of 10 to 20 people. They maintain a predominantly hunter-gatherer existence, are almost completely monolingual in their own language, and reject assimilation into mainstream Brazilian culture. The observations have also put into doubt the validity of the Chomskian theory of universal grammar and language acquisition in humans.
${ }^{12}$ Peter Gordon (2004) [24] conducted experiments to study Piraha counting and numeracy: "[Whilst] no language has been recorded that completely lacks number words, ... the counting system that differs perhaps most from our own is the 'one-two-many' system, where quantities beyond two are not counted but are simply referred to as 'many'. Members of the Piraha tribe use a 'one-two-many' system of counting. If a culture is limited to such a counting system, is it possible for its members to perceive or conceptualize quantities beyond limited sets, ... or to make what we consider to be quite trivial distinctions such as between four versus five objects? Without overt counting, humans and other animals possess an analog procedure whereby numerical quantities can be estimated with a limited degree of accuracy. When people use this analog estimation procedure, the variability of their estimates tends to increase as the target set size increases ... and can be indexed by a measure know as the coefficient of variation-the standard deviation of the estimates divided by set size. Although performance by the Piraha on [numeracy] tasks was quite poor for set sizes above two or three, it was not random. The value for the coefficient of variation is about the same as one finds in college students engaged in numerical estimation tasks. The results of these studies shows that the Piraha's impoverished counting system limits their ability to enumerate exact quantities when set sizes exceed two or three items. Participants showed evidence of using analog magnitude estimation. This split between exact enumeration ability for set sizes smaller than three and analog estimation for larger set sizes parallels findings from laboratory experiments with adults who are prevented from explicit counting [and in] studies of numerical abilities in prelinguistic infants, monkeys, birds, and rodents. The analog estimation abilities exhibited by the Piraha are a kind of numerical competence that appears to be [as with] lower animals, robust in the absence of language. [Can] humans who are not exposed to a number system represent exact quantities for medium-sized sets of four or five? The answer appears to be negative. The Piraha inherit just the abilities to exactly enumerate small sets of less than three items if processing factors are not unduly taxing. [It also suggests] how the possibly innate distinction between quantifying small versus large sets of objects is relatively unelaborated in a life without number words to capture those exact magnitudes. Numerical cognition is clearly affected by the lack of a counting system in the language."
${ }^{13}$ Dan Everett (2005) 15 explains why: "Piraha culture restricts communication to the immediate experience of the interlocutors [and precludes] abstract subjects. ... This cultural constraint explains a number of very surprising features of Piraha grammar and culture: the absence of numbers of any kind or a concept of counting and lack of any terms for quantification such as 'all', 'each', 'every', 'most', or 'some', the absence of color terms, the simplest pronoun inventory known, the simplest kinship system yet documented, the absence of creation myths and fiction, the absence of any individual or collective memory of more than two generations past, the absence of drawing or other art and one of the simplest material cultures documented, and the fact that the Piraha are monolingual after more than 200 years of regular contact with the Brazilians and the Tupi-Guarani speaking Kawahiv.
${ }^{14}$ It has been made clear by all who lived amongst and conducted research on the Piraha that the observations are not due to limited mental faculties or primitiveness. Everett writes: "No one should draw the conclusion that the Piraha language is in any way 'primitive'. It has the most complex verbal morphology I am aware of and a strikingly complex prosodic system. Gordon concurs: "One can safely rule out that the Piraha are mentally retarded. Their hunting, spatial, categorization, and linguistic skills are remarkable, and they show no clinical signs of retardation." Everett writes further: "The Piraha are some of the brightest, pleasantest, most fun-loving people that I know. The absence of formal fiction, myths, etc., does not mean that they do not or cannot joke or lie." A captivating 8-minute audio in which Everett narrates his life among the Piraha gives a sense of their thought processes ("Losing Religion to the Piraha", https://www.youtube.com/watch?v=rKqxnU5P1Yc). The 12,000 word article in the New Yorker gives additional insight into this unique culture. [7]
${ }^{15}$ Prior to this, mathematical historians (Ginsburg, others), assumed the number sense was universal and not cultural. [25], [28], [?], et al.
${ }^{16}$ For Sumeria and Mesopotamia, the impetus appears to have been statal administration with religious obligations towards justice and redistribution of wealth [26. Investigation shows that all leading civilizations, ancient and modern (Babylonian, Egyptian, Incan, Mayan, Aztec, Hindu, Chinese, Greek, Arabic, European) have reached a high degree of numerical facility. Those aboriginal cultures which, while not developing numeracy indiginously, are still able, when the opportunity arises, to produce numerate individuals and absorb numerate elements into their lives. In this context, the Piraha culture is profoundly unique given both its longevity (200 years contact with the Portugese) and its successfully maintained conscious aversion to accruing anything abstract-whether art, fiction, planning, farming, herding, or even counting.
${ }^{17}$ borrowing, lending, and interest payments
${ }^{18}$ By 3000 BCE , the Babylonians were doing arithmetic, and by 2500 BCE had a rudimentary abacus and were able to calculate square and cube roots, knew the Pythagorean theoream thousands of years before Pythagoras, and could handle an astonishing number of complex problems using a sophisticated place value system in base 60 (judiciously chosen as the smallest number that has all divisors through 6). They had a complex system of measurements, organized administrators, collected taxes and levies, distributed wealth, had a scribal system with schools and students as well as a temple culture of research and learning, from which they developed an advanced scientific understanding of astronomy and the calendar http://www.storyofmathematics.com/sumerian.html, [?], [26, 14, http://www.thocp.net/timeline/0000.htm, https://en.wikipedia.org/wiki/History_of_mathematics Appendix 2. It is also interesting to note that negative numbers would later be viewed in Europe with the same discomfort as imaginary numbers.
${ }^{19}$ astronomy, also required for celestial navigation
${ }^{20}$ The distillation of this collective numerical experience goes through varying degrees of loss in content, capability, and coherence in its journey to school-books. The arrangement between a shared legacy and a viable curriculum is admittedly
an uneasy one. Doing it well sows the seeds for later discovery of the crucial connections between number, space, time, and motion and the production of future scientists.
${ }^{21}$ In Hoyrup, we see clearly how the expansion of administrative functions in early Sumerian city-state structure led from the use of tokens and counters as metrological symbols to the gradual development of genuine mathematics, and the abstraction of number, with applications to the calendar. This occurred in the context of the temple systems, with a scribal culture that had two streams: the accounting stream and the surveying stream. Scribes were taught in schools, with mathematical problems on tablets. 39. "The formation of mathematics [3,000 B.C.E.-2,000 B.C.E.] as a relatively coherent complex was thus concomitant with the unfolding of the specific Uruk state... . Bureaucracy itself does not demand the type of coherence inherent in the Uruk formation of mathematics. What is involved is, a particular spirit of bureaucracy, one tempted by intellectual-and not by merely bureaucratic-order" (p.42-43) (Note: Uruk was the ancient city of Gilgamesh.) 26, 39]
${ }^{22}$ Through banking and interest rates one gets to the transcendental $e$ in the limit, fixing the calendar leads to astronomy which leads to trigonometry, another path to the transcendentals. Geometrical measures lead to the algebraic irrationals, metrology to duplication of the cube and solvability by radicals, and curvature to the conic sections, quadratics, and imaginary roots.
${ }^{23}$ An irrational number therefore must have an infinite and non-repeating decimal expansion.
${ }^{24}$ FAQ9. It is worth noting that computational algebra systems (CAS) are able to work entirely in rationals. How do they represent the entire set, both the very large and very small? How are they represented under-the-hood in finite precision (e.g. 64-bit)?
${ }^{25} \mathrm{FAQ}$ : floating point numbers and their error analysis. Describe for an 8 -bit, 16 -bit, and 64 -bit computer.
${ }^{26}$ FAQ1: Infinite sets are problematic, even countably infinite ones such as the whole numbers, as they require second order logic for their construction (first order logic is only capable of working with finite sets). And while a finite set with only addition can escape Godels' incompleteness theorem, infinite set and two operator arithmetic implies a logic structure that is incomplete. FAQ2: development and improvement of these numerical methods.
${ }^{27}$ Simon Plouffe discovered the BBP algorithm and used it to obtain the $n$-th digit of $\pi$ in 1995 (https://en.wikipedia.org/wiki/Simon_Plouffe).
Fabric Bellard improved upon this in 1997. (https://en.wikipedia.org/wiki/Fabrice_Bellard His software and mathematical notes are here: http://bellard.org/
${ }^{28}$ These are irrational because the elliptic integrals 38 involve roots, which we know are irrational.
${ }^{29}$ Indeed, this means this simple equation is insoluble for most composite numbers. It's actually rather unusual to have a perfect square. While there are infinitely many of these, they are ever more sparsely distributed in $\mathbb{N}$.
${ }^{30}$ FAQ: Closure is an algebraic property. The rationals are a field. It will turn out that the largest reasonable extensions are at least a division algebra, of which there are only 4 of finite dimension, $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. See Division Algebra: https://en.wikipedia.org/wiki/Division_algebra, Algebra over a field, https://en.wikipedia.org/wiki/Algebra_over_a_field, Algebraically closed field, https://en.wikipedia.org/wiki/Algebraically_closed_field
${ }^{31}$ majority of real numbers are non-computable in the sense, e.g. of Specker sequences, which don't have a computable supremum despite being bounded and strictly increasing
${ }^{32}$ Definable number: https://en.wikipedia.org/wiki/Definable_real_number Arithmetic closure is automatic by definition. Are there any non-definable numbers?
${ }^{33}$ Since to know such a number is to be able to specify it with a definition
${ }^{34}$ Mathematical constants: https://en.wikipedia.org/wiki/Mathematical_constant
${ }^{35}$ This is because by constructive methods we could only reach countable infinity (recall even the computable and definable numbers obtained above were countably infinite).
${ }^{36}$ Connectedness as a concept relies on continuity. Does connectedness require the axiom that the unit closed $[0,1]$ interval is connected? Does continuity require the reals-i.e. can it be defined on the rationals? What about functions defined solely on the rationals, and irrationals, e.g. $f(x)=0$ if rational and 1 if irrational. Such a function is clearly not continuous everywhere, but is it continuous anywhere? This has to do with the density of the rationals and irrationals in the reals. Why is a discrete set disconnected?
${ }^{37}$ Is it acceptable from a set theory perspective to consider ll' such sequences? Is this correct - $\mathbb{R}$ are complete by definition? Revisit Rudin.
${ }^{38}$ It is a connected set in the sense of having a mathematically continuous path between every pair of points. The definition of continuity is itself based on the properties of the rationals and the axiom that the interval $(0,1)$ is connected ensures that there is no hole, no minimum sized gap in the set of numbers. Implicitly we are using a distance measure, a metric, relating distance between points with coordinates, numbers.
${ }^{39}$ Looking at Chaitin's constant, one could make an argument for undefinable numbers being analogous to decision problems based on free will - e.g. binary encoding what a robot would do at all decision points in an infinite maze (or for a person, all decision points they face in life), and potentially all of the knock-on possibilities (considering all branches).
${ }^{40}$ See Lakatov for discussion of monsters in mathematics. See counterexamples in analysis and topology for a catalog of these. See remarks.
${ }^{41}$ See Solomon Feferman
${ }^{42}$ which takes precedence, $1+i$, or $1-i$ ? Working in the reals saw branch cuts occur when taking the inverse of a one-tomany function such as $x^{2}$ or $\sin (x)$, and by convention a primary or canonical branch. In $\mathbb{C}$, branching occurs much more frequently, as all elementary functions (logarithmic, exponential, trigonometric, and hyperbolic) are one-to-many. (In $\mathbb{R}, \log$ and exp were one-to-one.).
${ }^{43}$ The Argand diagram shows complex numbers as vectors. The Euler identity and its derivation explains why multiplication in $\mathbb{C}$ is expansion and rotation:
$$
z=x+i y=r \operatorname{cis}(\theta)=\cos (\theta)+i \sin (\theta)=r e^{i \theta}
$$
by recognizing $x, y$ as projections in 2-D argand space, then replacing cos and sin with their infinite series expansions and comparing with the series expansion of exp formally evaluated at $i$ (FAQ6)/
${ }^{44}$ See John Baez.
${ }^{45}$ It's actually quite remarkable and a significant constraint to require a set with two operations, an identity for both, inverses for both, and the fact that there are no zero divisors. Why do the sedenions (the next algebra above octonions) lose divisibility?
${ }^{46}$ There are two essential elements in the number concept: 1) the possession of an algebraic structure, and 2) the correspondence between numbers, points, and lengths. Generalizations then occur algebraically for objects with closed binary operations (e.g. a group), and geometrically/analytically for points having a geometric structure (e.g. vectors, Lie algebras).
${ }^{47}$ The view of complex numbers as inducing motions is the perspective of conformal mapping in complex variables theory.
${ }^{48}$ Noether's 1921 paper, "Ideal Theory in Rings" is considered as the tipping point to the abstract point of view: "All relations between numbers, functions and operations become perspicuous, capable of generalization, and truly fruitful after being detached from specific examples, and traced back to conceptual connections." (Noether). 33 ,p.91. The abstract direction of algebra was spreading since the 1880s as results known in their concrete setting from a century earlier (starting in the 1770s) with the work of Lagrange, Abel, Galois, and Cauchy, were now reset into a more abstract setting and expanded in the work of Jordan, Holder, Klein, Lie, Cayley, Frobenius, and Dedekind) 32, p.211. Lagrange and Abel had studied the symmetric group on polynomial roots, while Galois used group theory to solve the quintic problem (showing it was insoluble). Hamilton discovered the quaternions in 1845, and Graves the Octonions-the first non-associative algebra—in 1847 with Cayley discovering it independently not long thereafter. The formalization of abstract algebra strongly accelerated from 1921 to proliferate into many specializations in the mid and late 20th century. Of the period from 1770-1880, Bell wrote (1945, The Development of Mathematics): "The entire development required about a century. Its progress is typical of the evolution of any major mathematical discipline of the recent period; first, the discovery of isolated phenomena, then the recognition of certain features common to all, next the search for further instances, their detailed calculation and classification; then the emergence of general principles making further calculations superflous, unless needed for some definite application; and last, the formulation of postulates crystallizing in abstract form the structure of the system investigated." 32], p. 207
${ }^{49}$ Dedekind first used $\mathbb{N}$ in 1888 to denote the positive integers, or 'naturals'.
${ }^{50}$ Bourbaki standardized in the 1930 s the symbol $\mathbb{Z}$ for integers meaning 'Zahlen' or 'number' in German.
${ }^{51}$ Note: Gaussian integers, i.e. $a+b i \in \mathbb{C}$ with $a, b \in \mathbb{Z}$ form a ring (exercise).
${ }^{52}$ Bourbaki standardized in the 1930s the symbol for integers meaning 'quotient' in German http://jeff560.tripod.com/nth.html https://en.wikipedia.org/wiki/History_of_mathematical_notation
${ }^{53}$ aka skew field. http://math.stackexchange.com/questions/331020/do-people-ever-study-non-commutative-fields
${ }^{54}$ Rock $\cdot$ Scissors $=$ Rock; Rock $\cdot$ paper $=$ Paper; Scissors $\cdot$ Paper $=$ Scissors. Note commutativity. But not associativity: (Rock paper) • Scissors $=$ Paper $\cdot$ Scissors $=$ Scissors $\neq$ Rock $=$ Rock $\cdot$ Scissors $=$ Rock $\cdot$ (Paper Scissors). https://www.tofugu.com/japan/japanese-rock-paper-scissors/ Tournaments are another example of a non-associative algebra: A beats $\mathrm{B}, \mathrm{B}$ beats C , but C beats A . One might even say that life in general is non-associative because of the complexity of how different talents interact. So associativity holds when $A, B, C$ are simple, single-attribute objects, or when there is no element of chance. Voting is another example. The cycles resemble warfare: cavalry effective against archers, archers against pikemen, and pikemen against cavalry. There are connections with game theory, non-linear dynamics, graph theory (for extensions with more elements), parity algorithms, and automated game playing algorithms which do historical analysis) https://en.wikipedia.org/wiki/Rock-paper-scissors
${ }^{55}$ One has fourth order differential equations for which the characteristic equation is a fourth degree polynomial.

## Recommended Reading.

(1) 44 covers much of the material in this article in greater depth by a first rate research mathematician and prize winning expository writer of mathematics.
(2) [17] covers the foundations of our number systems in exceptional clarity by one of the foremost logicians and authorities on the subject.
(3) 34 covers the construction of the number systems in an ultra-dry, Bourbaki style.
(4) [36] covers everything you wanted to know about algebraic and transcendental numbers - it is the monograph I would have liked to have written.
(5) [6] is an outstanding expository article on the octonions and their connections to higher algebra and geometry.
(6) [27] is an excellent expository article on the applications of the theory of octonions to string theory.
(7) [43] is another exposition that covers the quaternions and octonions and their colorful history, written by a master expositor.
(8) [33] outstanding history of algebra and how it went from Greek constructions to Galois theory and the modern perspective in mathematics.
(9) 10 covers the history of the development of vectors in algebra, from Leibniz searching for a universal language of logic and geometry, to Grassmanians the battle between Hamilton's quaternions and the vectors of Gibbs and Heaviside (physics and engineering).
(10) [37] covers the history of how Astronomy stimulated the development of trigonometry, number system, and calculations.
(11) Tevian Gray's expository papers on the Octonions. http://people.oregonstate.edu//drayt/cv/recent.html

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[8] Richard Courant. What is mathematics? (an article). 1941. This is the first five pages of the book of the same name 9]. Essay is available at http://www.math.uga.edu/~azoff/ffds.pdf Article includes an explanation of how the viewpoint of modern mathematics is the same as that of modern science, and how this has necessarily led to the axiomatic / postulational formulation of mathematics.
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and mathematical culture. It presents in an accessible manner, in one place, a well-structured and tantalizing journey of progress through mathematical history. Highly recommended. Recommend [28] as a more detailed second reference on the history of numbers.
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## 1. Reflections and Historical Remarks

This period of mathematics is fascinating for its intersection between philosophers (Wittgenstein, Berkeley, Feferman) and its great mathematicians: Bolzano, Weierstrass, Cauchy, Hilbert, Dedekind, and Kroneckar, etc. ${ }^{56}$ The intensity of the debate can be surmised from this comment by Hilbert, a towering figure in 19th century mathematics: "We shall not be thrown out from this paradise with Cantor has created for us." ${ }^{57}$

What paradise is this? A way out of the bind of not having a model for the continuum, i.e. a continuous, connected space, consisting of distinct points, but which can be put into one-to-one correspondence with number and not have any holes (by definition it contains all possible convergent sequences.)

Interestingly, modern physics has shown that space and time are inseparable (Minkowski spacetime) but also that they are discrete not continuous-Planck distance and Planck time define the quantum limits. (Recall the famous metaphysical debates among pre-Socratic Greek philosophers on the nature of reality, space and time. $)^{58}$

Physical reality notwithstanding, the continuum is a highly desirable model for mathematics as it simplifies questions such as 'does a line pass through every point?', 'do two non-parallel lines actually intersect in the same point?'

Cantor's results upended deeply held beliefs about the importance of dimension in determining the size of spaces. He showed that there is no difference in the cardinality of $\mathbb{R}$ and $\mathbb{R}^{n}$, for which the famous space filling curve of Peano, and other paradoxical constructions, provide examples.

The vastness in the real number system of numbers which can neither be defined nor used, is analogous to dark matter in the universe. ${ }^{59}$ (One observes a similarity between plots of algebraic numbers in $\mathbb{C}$ and a sky map of the stars. ${ }^{60}$ ) Once a single actual infinity is admitted, even $\mathbb{N}$, then in a logical sense the passage to higher orders of infinity is unavoidable, as every set carries with it the concept of the power set (set of all subsets), creating a hierarchy of infinite cardinalities. ${ }^{61}$ Hermann Weyl writes (finitist point of view):

> "classical logic was abstracted from the mathematics of finite sets and their subsets. Forgetful of this limited origin, one afterwards mistook that logic [e.g. power set operation] for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and original sin of [Cantor's] set theory."

The key step is the power set operation, beginning with the power set of the naturals to obtain the reals, thence the power set of the reals, and the power set of the power set, etc. in an unbounded chain of higher order infinities.

Having once entered Cantor's world (through the axiom of infinity), we are assured that any number that we may find or any convergent sequence that is created, is included by definition within the continuum. The cost of this assurance is the increase vigilance required to suitably qualify all analytics results to exclude the wild world of counter-examples.

Are the Reals really real? Brillouin has argued against the use of the real numbers. I offer a different argument against them. If all definable numbers are countably infinite, then the uncountable infinity of the reals are superfluous numerically. They are needed to ensure the continuum. But physics suggests the real world is not a continuum but rather has quanta. So constructing a continuum is a nice logical exercise but does not represent (our) reality. And once one allows the creation of one higher infinity, then why not the others. Now, on the other hand, one can argue (as Weyl) that the moment one allows the countably infinite to exist in actuality rather than as a potential infinity, never realized, then one becomes a victim of the logical necessity of the powerset operations, using which one gets immediately beyond the countable infinity to the higher orders of infinity. The costs of using $\mathbb{R}$ are high - one has endless paradoxes and monsters in function theory etc., and one must spend a lot of time qualifying carefully ones arguments to avoid falling into a trip. What would break if one did not accept it, but rather stuck with the definable numbers?

The nature of reality. A number should mark location. We view this as coordinates. On a line, a single real number suffices (maps the continuum). In the plane, a complex number, or a vector in $\mathbb{R}^{2}$ suffices. In three space, it is a vector in $\mathbb{R}^{3}$, but this is not a division algebra and cannot be made into one. The best we have is the quaternions, a 4 -dimensional number (non-commutative). From physics, we know that it is not just space, but spacetime (as Minkowski says). Are the four dimensional quaternions related? From mathematics, we know that the only other division algebra possible is the 8 -dimensional octonions (non-associative). From physics, we know that reality may be 10 -dimensional, and that this is directly related to the 8 -dimensional octonions. All of this suggests that there is something really deep going on,
and that indeed, we are not there yet fully on what seems, on the surface of it, to be a very simple question - what is number?

## 2. Frequently Asked Questions

FAQ X: What's tbe big idea here? The notions of number, time, motion, and space have changed dramatically over the past ten millenia.(FAQ1,2) A gradual process of algebraic and analytic extension has unified them through a generalized concept of number-transforming modern mathematics with it.

Number: Number began as a description specific to a collection of objects (so 4 sheep and 4 goats are different) and only later came to describe the attribute of quantity (e.g. the 'four-ness' that they share in common). In the Near East (Turkey, Iraq, Iran, Palestine) the transition to abstract number was facilitated by the use of accounting tokens to represent the items. Abstract number allows arithmetic to become algorithmic which in turn makes the closure of number under its possible operations very important - indeed most developments in number have been extensions to gain operational closure under additional operations, from subtraction, to division, taking roots, and taking limits.

What happened over the ten millenia: c.8,000-4,000 BCE Neolithic cultures in Iraq, Iran, Palestine, and Turkey use clay tokens for record-keeping of accounts. c.4,000-3,000 in Sumer the transition to writing accompanying the rise of larger city-states with centralized statal administration and the need to archive accounts. First hollow clay envelopes containing the tokens with symbols on the outside summarizing the content, followed by the same symbols but on tablets without the tokens at all. "The token system reflected an archaic mode of 'concrete' counting prior to the invention of abstract numbers. There were no tokens for ' 1 ' or ' 10 '. Instead, a particular counter was needed to account for each type of goods: jars of oil were counted with ovoids, small measures of grain with cones, large measures of grain with spheres. Tokens were used in one-to-one correspondence: one jar of oil was shown by one ovoid, two jars of oil by two ovoids." 41, p.7., [26, p.33-36. Progress from tokens to tablets evolved in Susa (Elam/Iran) and in Uruk-V (Sumer) between (3500-3000 BCE) as the number of commodities being tracked increased and these themselves needed ever greater detail (e.g. sheep distinguished into ewe, ram, lamb), and as economic goods began to be moved across a larger statal area (requiring delivery notes). [26],p.36. By 3100 BCE, there was a mathematics tradition in Uruk-III. By 2500 BCE (Early Dynastic III), the move to abstract numbers is completed, visible in the use of words to indicate the metric and numbers constant between the different types of things being counted. Now the base of representation was further abstracted and perfected for arithmetic (specifically multiplication and division) through the place decimal system with a standard base. [26],p. 45 Presumably base 60 was chosen for its many divisors: all of the first six whole numbers, and 10 overall $(2,3,4,5,6,10,12,15,20,30)$. To see this perfection, try doing mathematics with Roman numerals, specifically multiplication or division. From start to finish, the journey from proto-writing to abstract writing with arithmetic ability was accomplished in about 1000 years (3500-2500 BCE). From 2500 onwards we see the development of the Sumerian scribal culture, which continued into the Akkadian Empire (Sargon's expansion), and reached a high point in Ur-III under King Shulgi in 2100 BCE.[26],p.48. by 2500, we have reached numerical abstraction and place value representation of weights measure. By 2100, we have the high-point of Sumerian scribal mathematics in Ur III under King Shulgi, himself also trained in the scribal school, and therefore lierate: "When I was young I learned at school the scribal art on the tablets of Sumer and Akkad. Among the high-born no-one could write like me. Where people go for instruction in the scribal art there I mastered completely subtraction, addition, calculation and accounting." - Two Sulgi Hymns (Castellino, 1972) The first five millenia, from c.8000-c. 3000 BCE see the movement from clay tokens to proto-writing and the abstraction of the number concept from its specific qualities ( 4 sheep, 4 jars of oil) as they developed the accounting capability for centralized statal administration with a strong justice and redistribution function. The next five millenia ( 3000 BCE to present) saw the destruction of the Bronze Age, the reversion to creation myths within each civilization. For example, the four essential quantities, the birth of the world from one egg, or from duality-darkness and light, the cyclical nature of time (Hindu), the physical separation of the cosmos. And finally the gradual deepening of understanding, driven by researches mathematical, physical, and philosophical, which served to transform both mathematics and its applications.

Unification vs. previously: The Greeks shied away from number after discovering the incommensurability of a unit and the diagonal of a unit square, separating in their science space and length from whole number. Motion was geometric, through space. Fermat and Descartes returned number to space through the notion of coordinates that vary continuously. Cantor, Cauchy, Weierstrass created the analytical (arithmetic) justification for the reals. Newtonian conception of time was a single flow, linear, unchanging, constant. Minkowski and Einstein unified space and time into a single physical context. The complex numbers brought in rotation, continued through the quaternions. Abstract algbra has extracted the essential properties to allow any sort of computation, and group theory has captured motions in this. Through algebraic structures, the arithmetization of the reals, analytic geometry, and the topological concepts of continuity, connectedness, and completeness, number has now unified both motion and space.

FAQ X2: What is number? The key question is how general we can reasonably go in defining number. Not every set with a binary operation can be called number, since e.g. colors can be combined, but are not numbers. Same with rock paper scissors. So there is some notion of quantity that is distinct from set of elements that can be combined. For the latter there is the question of closed combination. E.g. chemical combination is not closed: elements can be combined, but their combination leads to ever more distinct materials with distinct properties. So what should define quantity? Is it the existence of inverses? This would eliminate colors and rock-paper-scissors. Does existence of inverses imply closure? So we still get groups.

Number implies some sense of quantity, with a (precise) sense of scale (bigger, smaller), of direction (more, less), of order at least in a single dimension (more along this line, less, more of this, less of that). So this means a,+- , and multiply at least.
E.g. $\mathbb{N}^{+}$are not even a group, even though they have two operations (that aren't closed). See FAQ 4a.

### 2.1. FAQ 1: What is Mathematics? One viewpoint-held by MacLane[35], Philip Davis and Reuben

 Hersh[11] (check Aleksandrov?) -is that mathematics is the science of number, time, motion, and space ${ }^{62}$Another - held by Russell, Hilbert, Peano, Frege - is that mathematics is reducible either to formalism (Hilbert) or to logical fundamentals ${ }^{63}$. While logicism has been shown not to be a tenable position thanks to Godel's groundbreaking work resulting in the Incompleteness Theorems of first order logic, Formalism is still a commonly held point of view. Courant cautions agains the narrow subscription to formalism in [8.

A third view-held by Courant [8] is that Mathematics is a humanistic science devoted to a deeper understanding of quantifiable phenomenon, in which concreteness and generality both have a part to play.

FAQ 1a. Is mathematics a science in the same way as physics, chemistry, or biology? Mathematics is a science because it proceeds by the scientific method (hypothesis, observation, conclusion). Falsifiable hypotheses are presented and either established or refuted (conclusion). But in the types observation, experiment, and proof (justification), it is different: mathematical truths may be obtained through observation and examples in an inductive manner similar to the other sciences. A mature subject in mathematics typically has its knowledge organized into an axiomatic framework, after which a particular statement is either valid or not. The refutation of a valid statement is then essentially the rejection of the entire axiom system as a whole or of a particular axiom.

FAQ 1b. Doesn't physics also study time, motion, and space? (What is the difference between mathematics and physics?) Time, motion, and space are indeed studied by physics, though the physics studies build upon the mathematical understanding of these. For example, progress in the understanding of spacetime in general relativity is built upon the mathematics of differential and riemannian manifolds and was reached through mathematics. Much earlier, progress in understanding motion was arrived at using physical experimentation with falling objects and rolling objects along inclines. But the understanding and characterization of motion was mathematical, and Galileo used functions to describe motion. The calculus of Newton and Leibniz describing the relationship between between rate and accumulation-derivative and integral-this was also mathematics. The solution of the differential equations of physics are solutions arrived at in mathematics even the equation itself arises out of modelling physical phenomena.

Mathematics and Physics are intimately connected. Physics is the study of force and energy - the things that cause movement and change - and the experimental verification of theories of matter, time, space, and the universe. But the descriptions of physics are mathematical, from mechanics to electromagnetism, relativity to quantum mechanics and particle physics. Wherever the fields have gotten too far apart, it has been to detriment of mathematics - as it means that it was the physicists who were doing the mathematics while the attention of the mathematicians was elsewhere occupied. ${ }^{64}$

FAQ 1c. What is the mathematics of Time? Time as conceived by Newton is a continuous linear flow, constant, never changing. With the General Theory of Relativity, we have now a different understanding of what time really is in our universe, based on a four-dimensional curved manifold call spacetime, yet the mathematical model of abstract time is still Newtonian. Both understandings require the theory of the continuum, which takes the concept of number beyond the rationals to the reals. This is necessary because the rationals, while not discrete (there is no minimum gap between rational numbers) are yet not complete or connected-indeed they are a totally disconnected set-while the reals are both complete and connected. They are measure 0 , so in the technical sense of 'almost everywhere' one could say that a
function $f$ that is 1 on the irrationals and 0 on the rationals is constant almost everywhere and continuous almost everywhere. Can one define continuity on the rationals?

FAQ 1d. What is the mathematics of Motion? Motion takes the notion of time and relates it to change through rates. This includes the differential and integral calculus for Newtonian mechanics, and the Lorentz metric and differential and riemannian geometry for spacetime general relativity. The calculus was a watershed as it had profound applications in science through differential equations and numerical and computational mathematics. It also includes the theory of geometric transformation, incluing rigid motions, contractions, conformal mappings, fixed point theory, algebraic topology (on orbits), and other areas.

FAQ 1e. What is the mathematics of Space? refers to both the setting in which we believe we live (the spacetime fabric of Minkowski ${ }^{65}$ ), and the various models of space ranging from the linear Euclidean space of a flat world, to the differential geometry of Gauss, to the various alternative spaces possible (classified through homology in algebraic and differential topology). However, fundamentally, the simplest space is that of number itself, whether this is $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or $\mathbb{R}^{n}$.

FAQ 1f. Why is time used to motivate the real numbers, not space, contrary to history? For two reasons: 1) space is now not taken by assumption to be Euclidean, so that the length of the diagonal of a square is in fact dependent on the curvature of the surface into which it is embedded; 2) it is questionable in fact whether space is in fact continuous, or rather whether there exists some discrete quantum of space, at the scale of Planck distance. Thus, a view of space that is continuous becomes the model for the reals, or vice versa, the reals model a particular view of one-dimensional space. So in that sense, we do not want the reals to tell us something about space, but rather our researches on space lead to the reals as a particular model. Time, by contrast, is about moments, and while physics also proposes a quantum of time at the scale of Planck time, it is easier to think of the continuous flow of time than the continuous fabric of space. The bottom line is that space is no longer as much of an axiomatic presupposition, while time relatively is still perceived that way. Motion is closer to number and time - it is about rate of change. So the calculus can proceed with just number, time, and motion, without requiring space. Space, on the other hand, is much more complex, as mathematical and physical researches have shown.

FAQ 1g. Does Motion require Space? Yes in the sense that motion implies change and change implies time and time requires number and number is the simplest space (FAQ1e).

FAQ 1h. Can we say Mathematics is the study of Number, and still cover all of mathematics? Yes with the extended notion of number, since this includes space, time, and the description of change (whether through functions or groups - which are generalizations of number). Indeed the deep and thorough study of number takes one through almost every area of mathematics:

- arithmetic and algebra
- plane geometry (circles, triangles, squares, their lengths and areas)
- elementary number theory
- trigonometric functions
- continued fractions
- Transcendental numbers
- approximations
- real analysis
- polynomials
- complex numbers
- infinite series
- limits and sequences
- taylor series
- Elliptic integrals and elliptic functions
- Bessel and hypergeometric functions
- floating point numbers
- Set theory
- Logic
- Group theory
- Abstract algebra
- Galois theory
- Topology (as an extension of metric spaces)
- Linear algebra
- Vector geometry
- Quaternion geometry
- Octonions and string theory

FAQ 1i. Is mathematics the science also of relations? Relations include order, equality, equivalence, one-to-one, many-to-one (function), one-to-many (set relation), partition (mutually exclusive, collectively exhaustive).

While relations are everywhere, and mathematics has systematized them and classified them and developed methods for working with them, it is not the point of the subject. Just as the point of zoology is to study animals, not to study taxonomies, though taxonomical classification is key, so the point of mathematics as a subject is not to study functions per se, even though a subfield of mathematics may indeed choose to do so, for a particular reason - and those functions will be of interest for a particular reason.

FAQ 1j. What do mathematicians do?

- Count (enumerate), calculate, estimate.
- Model, predict, control.
- Theorize, reason, prove.
- Explain, write, teach.
- Explore, experiment, program.
- Define, construct, collect (example \& counter-examples).
- Simplify, generalize, extract.
- Organize (structure), arrange (sequence/order), refine.
- Analyze (deconstruct), synthesize (put together), unify.

FAQ 1k. How can numbers be used in different ways to get so many different meanings? A good example is statistics: a number can mean a count of items, a percentage (ratio), a median (or middle value), a mean (or average) calculated any number of ways, a max or min, an ordinal value (e.g. the 2nd or third highest), a quantile (75th percentile value, etc.), or a measure of dispersion (standard deviation, coefficient of variation, interquartile range).

Coefficient of variation is $\sigma / \mu$ and gives the $\%$ variation of the data around the mean (no assumptions made about the symmetry of the distribution - this has to be checked). Note calculations can be made even when the context is wrong, e.g. one can calculate a tiny standard deviation even if the distribution is symmetric bimodal but the two humps are far apart.

### 2.2. FAQ 2: Mathematical Evolution.

FAQ 2a. What are the main areas of mathematics? Analysis, geometry, abstract algebra, topology (big 4 pure mathematics subfields), statistics, mathematical modelling, computational mathematics, physics (big 4 applied mathematics subfields).

FAQ 2b. What is the difference between pure and applied mathematics? All mathematics is applied - pure mathematics is the application of mathematics to itself. (CHECK: is this from Davis and Hersh?).

FAQ 2c. What is the difference between discrete and continuous mathematics? Between discrete and continuous sets? Discrete sets consist entirely of isolated points ${ }^{66}$, i.e. sets for which one can find a small enough neighborhood so that only one point of the set is contained in it - the intuition is that the points are spread apart with a minimum distance between points.

Discrete mathematics is that which is done using *finite* methods typically using just the integers (e.g. combinatorics, elementary number theory) or at most a finite subset of the rationals, (e.g. discrete probability theory).

The reason why discrete probability can use the rationals and still be considered discrete is because the set of events being considered are finite, so you are only ever calling upon a finite number of rationals to describe them. (FAQ5)

FAQ 2c. What were the watershed moments in the development of number, and that influenced mathematical concepts more generally? The development of writing in Babylon 5000 years ago ( 3000 BCE ) and its use in recording numbers, the Babylonian development of base 60 for large astronomical and the place value system [?], the Greek discovery of irrationals and systematization of Euclidean geometry using axiomatization, the understanding of paradoxes related to the infinitely small[?], and a consideration of the infinite counting numbers[3], the use of transcendentals (trigonometry) to calculate astronomical motions and events, the development of the calculus and the fundamental theorem of calculus, the solution of algebraic equations - up to fifth degree, the repeated attempts to solve the impossible three geometric problems - duplication of cube, trisection of angle, squaring of circle, the use by Galois of groups of permutations to show the impossibility of solution by radicals approach to solving the general algebraic problem (degree 5 or higher), the researches by Cantor into higher orders of infinity, including the uncountability of $\mathbb{R}$, the foundation of analysis, the birth of point-set topology, the discovery of non-Euclidean geometry, the creation of algebraic topology and the classification of spaces, differential geometry of curves and surfaces.

FAQ 2d. How much mathematics was there before the permanent historical record? One has to assume that mathematical knowledge was present long before it was captured in a form which has been preserved and survived through the millenia to today. For example, what was written on clay tablets could previously have been scratched out on a sand-board. We know that Archimedes himself was killed by a Roman soldier as he was concentrating on the ground where he had drawn figures with a stick. This behavior observed in Syracuse long after the Babylonians, shows that in all probability, mathematics was known well before we find it written on tablets. Figures drawn on sand would not have survived. Only the most important would have been carve into, e.g. the ishango bone, suggesting that it was indeed a prime table that someone (early mathematician?) wished to preserve.

FAQ 2e. How has the mathematical notion of rigor vs. application evolved? Mathematical history shows that both discoverers and systematizers, problem-solvers and organizers, researches at the frontier and exposition of what is known, are all important, indeed vital, to the healthy development of mathematics. Indeed this ebb and flow of energy between expansion and systematization is found in especially sharp constrast several times in mathematics: in the Greek school of Euclid (4th century BCE), the French school of Cauchy (18th century), under the influence of the algebraists (Klein, Erlanger, Lie) and formalists (Hilbert) in Germany (19th century), and most recently again in the French reorganization of mathematics along the lines of Bourbaki (1950s). See Courant, Kleiner.

FAQ 2f. Are there human languages without any concept of number? One has been found - the Piraha language of the Piraha tribe deep in the Brazilian Amazon. Unique among indiginous tribes has been the extent that the Piraha culture has resisted integration with the outside world. They are entirely
monolingual, reject all outside technology, do not intermingle, and have maintained a consistent culture for at least hundreds of years, and possibly thousands of years. Sources.

FAQ 2g. How are we able to read ancient writing? The keys have been fortuitous finds of decrees or proclamations made by powerful kings ruling over large multi-lingual empires and requiring therefore the issuing of multi-lingual documents (think European Union today). Deciperhing Egyptian hieroglyphics came from the Rosetta stone discovery which also included the Greek translation, from which the hieroglyphs were deciphered. For Babylonian cuneiform writing, the key is the trilingual Behistun tablet depicting the glory of Darius the Great in Old Persian, Elamite, and Akkadian (Babylonian). Old Persion had just 43 signs and had been worked on since the early 1800s. This provided then, slowly, the key to understanding Akkadian.*

FAQ 2h: How did civilization rise out of the Stone Age? The transition from paleolithic (stone age) to neolithic (settled) populations happened gradually, with the earliest known transitions between 10,000-8,000 BCE in Sumer (Mesopotamia). (Lith=Stone, Paleo=Old, Neo=New). Neolithic civilizations abandoned the hunter-gatherer culture of the Paleolithic, built settlements, domesticated wheats and other grains, started farming, and domesticated animals - the dog domesticated in Mesopotamia as early as $10,000 \mathrm{BCE}^{\dagger}$, with the sheep and goat from $9,000 \mathrm{BCE}$ onwards, the ox from $7,000 \mathrm{BCE}$, and animal labor in the fields (plow) from 4,000 BCE. Materials up to Neolithic times were stone (flint), wood, bone, and sinew.

The Copper Age (Chalcolithic, Chalco=Copper, Lith=Stone) emerged from the Stone Age between $6,00-5,000$ BCE in Sumeria, and saw both copper ( Cu ) and natural materials (stone, wood, bone, sinew) used in parallel. Metal was worked (first hammering, then later annealing or heating, casting, and then smelting) into tools, weapons, and jewellry. Arrows and spears were now copper tipped instead of stone tipped. Axes and hoes, knives and blades made agriculture, cutting and fashioning easier and more precise, and allowed the felling of larger trees to create larger fields and sturdier housing. $\ddagger$ Shields and helmets could be fashioned for better protection. War now became safer (for those with weapons) and created social hierarchies: warriors on top, slaves from those conquered, farmers (as before), and traders, who brought ore from afar (ore was not in abundance in Mesopotamia, but fuels were - bitumin, naturally occuring petroleum, and metallurgy appears considerably advanced in Sumeria. ${ }^{\S}$

The Bronze Age, beginning c.3,200 BCE, was ushered in by the discovery that tin ( Sn ) smelted with copper ( $5 \%-10 \%$ ) to create bronze (a metal alloy, or mixture) which is an exceedingly hard metal comparable in strength to steel and at the same time is easier to work with. This accelerated civilization but was also violent. The arrival of metal triggered a set of violent expansions leading to the first large Sumerian cities, and to standing armies and empires."

The transition to Iron Age began 1,100 BCE and equalized society, as iron ( Fe ) is found naturally everywhere, indeed by mass it is the most common element in the Earth's outer (and indeed inner) core.

FAQ 2i: What is the brief history of Mesopotamia? The Mesopotamian region is in modern-day between the Tigris and Euphrates rivers as they run into the Persian gulf. Initially in this area were the settled city-states of Sumer (southern Iraq) and Elam (Iran). Around Sumer and Akkad were the Elamites (Susa) and Guti to the East (Iran), the Assyrians to the North, and the Amorites to the Northwest. Sumer comprised X cities: Eridu (origin city on the shore, and the temple city), Ur, Uruk, Lagash, Girsu in southern Sumer and Nippur, Kish, Sippar and Eshnunna in norther Sumer. Akkad (Agade) near Kish would be founded by Sargon much later as a rival power that would in a short amount of time unify the entire area under the Akkadian Empire.

As can be seen from the Sumerian King List [29, kingship was violent ("smitten with weapons") and moved frequently between the rival but cooperating city-states of Sumer and Akkad until 2100 BCE when within 50 years, Babylon under Hammurabi will rise. ${ }^{67}$

By 2000/1800 BCE, two thousand years after the rise of the Sumerian-Akkadian empires, Babylon has arisen and under Hammurabi would control all of the Mesopotamian region, overcoming Larsa in the south,

[^3]Eshnunna in the east, Mari in the northwest, and Ashur (Assyria) in the northeast (another great power around Ashur and Ninevah). ${ }^{68}$ Recommended reference.\|

FAQ 2j. What do we know about Babylonian mathematics? Babylonian mathematics is a broad collective that glosses over the significant distinctions between four distinct civilizations that stretched over 4000 years: Sumerian (south Iraq on the shores of the Persian gulf at the mouths of the Tigris and Euphrates - the Fertile crescent between these two rivers), Akkadian (inland), Babylonian (which emerged from the centre), and Assyrian (from the north).

Clay tokens for numerical record-keeping date to about $8,000 \mathrm{BCE}$ in unbroken succession to 3,000 BCE, precursors to cuneiform writing (cf. Denise Schmandt-Besserat) which emerged from 4,000 BCE.

The Sumero-Akkdian period occurs from 4,000 BCE to 2,000 BCE, in which we have the great kings of Ur, Uruk, Kish, Lagash, Nippur, including the epic hero Gilgamesh of Uruk.

Third-Millenium mathematics is associated with the Ur III period and focuses on computational and utilitarian mathematics associated with the administrative scribal function (though with non-utilitarian additions from the Akkadian organization).

From 2,000 BCE onwards, we have the Old Babylonian (OB) mathematics from 1830-1530 BCE.
Then there is a dark period after the sacking of Babylon, and the inward development of the scribal culture. Finally, the Late Babylonian period (from c. 600 BCE onwards) sees mathematics again arise in the form of priests now interested in astrology and not metrology.

FAQ 2k. How did writing come about? From the accounting and record-keeping needs of a settled society in which labor specialization and hierarchical organization had already begun, and which was large enough that kinship bonds were not the dominating relationship. (In paleolithic or aboriginal huntergatherer cultures, I would expect that clans are typically small, six to a dozen families, enough for the men to form a reasonable hunting party and pool sufficient labor for shelter, and maintain an adequate deterrant against attack, but not so large that shelter, wood, water, or food would be.(confirm)) Around 8,000 BCE in Neolithic cultures across the Near East we find small clay objects of various shapes (cones, spheres, disks, cylinders, etc.) used for accounting. With the growing bureaucratic structures, there was an administrative need to archive these accounts. This was done initially by placing summary markings on hollow clay envelopes containing the tokens, which required the use of symbols to signify the objects being accounted for. This evolving directly to make the same markings on tablets without the need for the tokens themselves, accompanied by addition of further symbols with agreed meaning. [41],p. 7

PriorThe earliest explanations were mythological or religious: that a full-fledged script had been communicated to humans by divine revelation. Then came the pictographic theory of Warburton published in 1738 which held that writing evolved from picture writing, based on evidence from Chinese, Egyptian and Meso-American texts. The discovery of cuneiform tablets in large quantities from Mesopotamia in the early 1900s challenged this theory as cuneiform is not pictographic and no earlier forms were discovered. The current theory for the evolution of writing in Mesopotamia is that counting and tokens for accounting were the precursor for writing, which initially were summary markings on envelopes containing these tokens requiring the use of symbols to signify the objects being accounted for, evolving to markings on tablets without the need for the tokens and the further additions of symbols for meaning. (Did cuneiform ever develop to phonetic wriing? Does this theory apply to Egyptian and Chinese cultures or is it only application to Mesopotamian writing? How did alphabets come about and writing move to phonetics?)

[^4]
### 2.3. FAQ 3: Quaternions, Octonions and Modern Physics.

FAQ X. What is string theory and why is it being studied? String theory is an attempt to finish what Einstein spent the last part of his life trying to find: a 'theory of everything' that could tie together his theory of general relativity (which explains gravity) and quantum mechanics (which explains electromagnetism, the strong force, and the weak force). These two theories are not fully compatible in the pre-string theory understanding of physics. String theory combines the two theories by assuming there are multiple dimensions beyond the four that we know.https://www.aps.org/careers/physicists/profiles/kaku.cfm

FAQ X. What is the history of string theory? From 1921 (Kaluza-Klein theory), 1943 (as Stheory), then either 1968 or 1970 to the present,https://en.wikipedia.org/wiki/History_of_string_theory Timeline: http://www.superstringtheory.com/history/history4.html

FAQ X. What suggested quaternions and octonions? Having algebraic fields in 1 and 2 dimensions suggests why not also in 3- or higher dimensions? It turns out that 3 is not possible, but 4 is, and 8 , but with each step you lose more and more of the algebraic structure, so by 16 you no finally lose division (zero divisors), and by $32 \ldots$ This was a logically and aesthetically exploration of possibility, the pursuit of a certain symmetry, and continuation of a thought experiment suggested by existing patterns, a "what-if" made real. Hamilton was in pursuit of higher dimensional possibilities suggested by lower dimensional observations (quaternions), with Graves and Cayley independently discovering the octonions. (Who discovered the sedenions and observed that for 16 and higher dimensions, one loses divisibility entirely - non zero zero divisors.

FAQ 3a. What is the difference between the Standard Model, a Grand Unified Theory (GUT) and a Theory of Everything (TOE)?. The Standard Model is what is currently accepted in particle physics. A Grand Unified Theory aims to unify into a single mathematical theory three of the main forces in the universe: strong, weak, and electromagnetic forces. A Theory of Everything unifies in addition, the fourth force: gravity. The Standard Model treats each force separately as a different theory, three gauge theories, and the general theory of relativity. https://en.wikipedia.org/wiki/Grand_Unified_Theory

FAQ 3a. What is a string? A string is a hypothesized fundamental particle from which all known particles of the universe and all known forces would be created. It is one-dimensional, like a curve or line segment, but oscillates so in time traces out a 2-dimensional surface.

FAQ 3b. Why are octonions the right way to describe the universe? The octonions are an 8 -dimensional division algebra, i.e. it is possible to do arithmetic, including (unique) division, within $\mathbb{O}$. Just as $\mathbb{C}$ are rotations and stretchings in the plane and $\mathbb{H}$ are the same in 3 -space, $\mathbb{O}$ are the same in 8 -dimensional space. Why is 8 -dimensional space somehow the right thing for the universe? The answer has to do with the Standard Model of physics. In order to describe the standard model mathematically, one needs spinors for matter particles and vectors for force particles, and these combine to create other particles. But if the combination were a group, i.e. a single set of numbers for both types, then they could be combined arithmetically. One would want the ability to divide, i.e. to have a division algebra. One can build such a model with any of the four division algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. But all models under 10dimensions have an anomaly of some sort or another - the only one that doesn't is the 10-dimensional one involving octonions and a string and time, and the 11-dimensional one involving octonions, a membrane (surface) and time. So, octonions currently appear to be the best model for describing supersymmetry theories (where the supersymmetric means that matter and force particles can be interchanged without changing the laws of physics) which are currently viewed as one approach to finding a 'theory of everything' (grand unified theory, or GUT). [4], [6], [5] In the mathematics of string theory, quantum gravity, and supersymmetry, mathematicians have an opportunity to really discover something profound about our universe (see Dyson's Missed Opportunities article [13]).

FAQ 3c. How are the octonions connected to string theory? Adding one dimension for a string (it's length?), and another for time (the string vibrates in time), then we need a 10 -dimensional space to describe the movement of any given string in time: it's length, and it's rotation in 8 -dimensions. If the fundamental particle of the universe is a 2-dimensional membrane rather than a string, then we need 11 -dimensions, but still the rotation in 8 -D space. Why 8-D space? It turns out that one can describe quantum gravity in any of the 4 division algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, because what is needed is in fact the
arithmetic closure of the division algebra itself. But only in the 10- or 11-dimensional case, with spinors or membranes, and rotations using octonions, do all the details work out (check what these are).

FAQ 3d. What makes a system non-associative? Why are the Octonions non-associative when the Quaternions are associative?

### 2.4. FAQ 4: Algebra.

FAQ 4a. Is there an hierarchy of algebraic structures? Two ways to traverse the hierarachy of algebraic structures: by starting with a set without structure and adding axioms, or starting with a rich, well-recognized structure, and removing axioms. If we start with a Group or Ring (e.g. $\mathbb{Z}$ under + or $+/ \mathrm{x}$ ), removing divisibility (inverses) gives a Monoid or Semiring(e.g. $\mathbb{N}$ ), removing identity gives a Semigroup (e.g. $\mathbb{N}^{+}$), removing closure gives a Semicategory (e.g. primes), replacing closure but removing associativity gives a Magma. Note in particular that an identify nor divisibility are not necessary for number combination, e.g. $\mathbb{N}^{+}$, and neither is closure, e.g. primes.

FAQ 4b. What are examples of non-commutativity and/or non-associativity in operations? (Add table, with examples). Commutative and Associative:,$+ \times$. Non-commutative and Non-Associative: ,$- \div$, and octonions. (see the non-associativity with $5,3,1$, an $12,6,2$ respectively). Not Commutative but Associative: Composition of functions, matrix products, quaternions. (see non-commutativity with $D_{4}$ symmetry group of a square, with rotation $r$ and reflection through vertical symmetry axis $v, r v \sim v r$, while $(r v) r=r(v r))$. Commutative but non-associative: Rock-paper-scissors ${ }^{69}$ Note, - is anti-commutative: same magnitude, opposite sign.

FAQ 4c. What is the language of arithmetic operations? Language is important because it indicates how to think about the operations mentally. (Add table, with examples). Plus + , Less - , Times, by (for areas), of (for multiplying by percents) $\times$, Into, per (for units) $\div$, Raise, Root $\sqrt[2]{ }$. "Six times three" means take three and repeat it six times for a total of 18. "Six into three" means take the six and divide it into three equal groups, making for two in each group.

FAQ 4d. What are examples of symmetric and asymmetric relations? Symmetric: Sibling, Partner Asymmetric: Brother/Sister, Husband/Wife, Parent/Child Friendship is an interesting one: in the short-term it can by asymmetric (I can be your friend but you not be mine.) In the long run, however, it is symmetric (at some point the non-reciprocity will be discovered by the friend, and the friendship will be severed - if not then the friendship is in deed reciprocal and therefore symmetrical).

FAQ 4e. How important is closure for algebra? It is not necessary for arithmetic or computation, but hugely important for algebra. For example: working in $\mathbb{N}$, one can calculate parts or subtract more than one has (considering it borrowing). However, because fractions and negative numbers are not part of number, all divisions and subtractions have to be qualified by: divisibility case and order of numbers. By expanding the number set to $\mathbb{Z}$ and then $\mathbb{Q}$ one does away with the need for qualifications - it's one equation that works for all cases. The same is true for working in $\mathbb{R}$ and $\mathbb{C}$ - convergence to an irrational number is no longer a special case, and neither is finding a zero to an algebraic equation through roots (at least for polynomials of degree less than five - see Galois FAQ).
which can be commutative, as in the case of Abelian groups e.g. $\mathbb{R}$ or $\mathbb{C}$, non-commutative, e.g. the composition operation and quaternions, or even non-associative, e.g. octonions

### 2.5. FAQ 5: Infinity.

FAQ 5a. How early was the infinitude of number sets $(\mathbb{N}, \mathbb{Z}, \mathbb{Q})$ known? On both the very large and the very small, Archimedes was familiar. In his appropriately titled 'Sand reckoner' (J.Newman's anthology) he describes how to continue naming numbers indefinitely (counting the grains of sand). On the small side, the principle of exhaustion calculated irrational quantities to arbitrary precision. Euclid knew about this when he showed that there are an infinitude of primes. Before him, the Greek philosophers were also aware of the paradoxes of the very large and very small (infinitesimals). Zeno knew about this when he explained the paradox of infinitesimal small times (Achilles and the Tortoise), and infinitesimally small distances and therefore infinite time to traverse a fixed length (Halfway to a Door).

Children learning to count implicitly assume that there is a stopping point, i.e. a biggest number, which they think of as a million, or a billion, or they just make up a very large sounding number. It comes as a surprise when one shows, yes but what about a million and one? then 10 million, 20 million, 100 milion, 900 million, then a billion, 10 billion, 20 billion, ..., 900 billion, trillion, and it goes on forever!

FAQ 5b. What is Cantor's Continuum Hypothesis, and how is related to the unbounded hierarchy of higher orders of infinity? Let $\aleph_{0}$ be the cardinality of $\mathbb{N}$. Let $2^{|S|}:=|\wp(S)|$, where $\wp$ is the powerset (set of all subsets) of set $S$. Define $\aleph_{1}=2^{\aleph_{0}}$, or the cardinality of the power set of the naturals. Recursively continuing this process, define $\aleph_{n}=2^{\aleph_{n-1}}$. Show that the power set $2^{S}$ can never be put into one-to-one correspondence with its parent set $S$ (even though an equivalent definition of an infinite set is that it can be put into correspondence with a strict subset ${ }^{70}$. So we have now an infinite hierarchy of increasing cardinalities.

There are now two questions: 1) where does the cardinality of the continuum ( $\mathbb{R}$ ), or $\aleph_{c}$, fit into this hierarachy? 2) is this hierarachy discrete and complete, i.e. is there any cardinality less in between the steps. In particular the second question asks whether there is a notion of continuum with a cardinality that is neither $\aleph_{0}$ nor $\aleph_{1}$.

Cantor's Continuum Hypothesis is that $\aleph_{c}=\aleph_{1}$. (See Feferman).
FAQ 5c. Why is it impossible to create the continuum without admitting the vastness of an infinite of higher order than countable? Cantor's diagonal argument shows there is no way to do this. To obtain the continuum, all possible gaps must be guaranteed to be filled. Essentially, we must have a set that is at least the cardinality of the power set of countable infinity, since it must include the set of all possible combinations of the integers. Once the concept of number has absorbed the rest of the continuum through completion, one no longer has gaps since every sequence of rationals converges.

FAQ 5d. What is the intuition for a continuous space? Can the rationals be a continuous space? Continuity makes precise the concept of an unbroken function tracing between spaces. The intuition of the unbroken space itself is the connected space.

The rationals, while not discrete (there is no minimum gap between rational numbers) are yet not complete or connected-indeed they are a totally disconnected set-while the reals are both complete and connected. They are measure 0 , so in the technical sense of 'almost everywhere' one could say that a function $f$ that is 1 on the irrationals and 0 on the rationals is constant almost everywhere and continuous almost everywhere. Can one define continuity on the rationals?

Discrete set: a set $S$ (embedded within in a topological space $X$ ) is discrete if it consists entirely of isolated pointshttps://en.wikipedia.org/wiki/Isolated_point $x$, around each of which a neighborhood can be found that contains no other points of the set. Examples: the Gaussian integers in the plane, the natural numbers in the reals. The rationals are not discrete because in every neighborhood of a rational lies another rational (this is the Archimedean property of rationals).

FAQ 5e. Are the rationals a discrete set? No, the rationals are not a discrete set even though they are countably infinite. This is because the rationals have no isolated points-you can always find a nearby rational number as close as you like. This is called the Archimedean property of the rationals, and you can see it by asking for any tiny fraction, say $1 / 100,000$, can I find a smaller one? Sure: $1 / 100,000,000$. So there is no neighborhood that you can put around a rational point that you could not find another rational within.

However, any *finite* collection of rational numbers *is* discrete since when you only have finitely many rationals, then there is a minimum distance between any two rationals in the set, so one can find an interval smaller than this which will guarantee only one rational is in it at any given time. This means a finite set of rationals is an isolated set, and therefore discrete.

FAQ 5f. What is sparse and dense in the reals? Are primes, integers sparse in the reals? Finite subsets of measure 0? Rationals and algebraic numbers dense but countably infinite? Irrationals dense as well, but uncountably infinite?

FAQ 5 g . How can we make precise our intuition that not all countably infinite sets are of the same 'size'? Similarly for the uncountable sets. E.g.
$\mid$ computables $|>|$ algebraics $|>|\mathbb{Q}|>|\mathbb{Z}|>|\mathbb{N}|>|2 \mathbb{N}|($ evens $)|>|n \mathbb{N}|, n>2>\mid$ primes $|>|$ squares $|>|$ cubes $\mid \ldots$ and similarly for the uncountable sets:
$\left|\mathbb{R}^{n}\right|, n>2,>|\mathbb{C}|>\left|\mathbb{R}^{2}\right|>|\mathbb{R}|>\mid$ transcendentals $|>|[0, n]|, n>1>|[0,1]|>|(0,1)|>|(0, \epsilon)|, 1>\epsilon>0, \ldots$
Cantor's arguments showed that under one-to-one correspondence (cardinality) there are only two classes of infinity - countable and uncountable - with no size distinction between any sets within the same class.

Measurability (Lesbesgue) makes precise the geometric notion of length of a set, so distinguishes between bounded uncountable subsets, e.g. $m([0, n])=n>m([0,1])=1$. But all countable sets are of measure 0 .

Cardinality distinguishes between finite sets - the usual set size and counts.
So the delicate sizing is with the countable family. Here we can consider frequentist or statistical notions. For example, what is the likelihood of encountering a prime between 10,000 and 100,000 , versus an even number, a multiple of $n$, a perfect square, perfect cube, etc.

FAQ 5h. What is Cantor's diagonal argument? The diagonal argument shows $\mathbb{R}$ cannot be countable by showing that a subset of $[0,1] \subset \mathbb{R}$ is not countable, which means it (and all sets containing it) must be uncountably infinite.

Proof Sketch: Suppose not, i.e. suppose $[0,1]$ were countable. Then all elements $x \in[0,1]$ could be put in 1-1 correspondence with the counting numbers, i.e. listed (enumerated). Consider a subset of such list containing the set $\left\{x_{1}, x_{2}, \ldots, x_{n}, \ldots\right\}$ of all decimals whose expansions only use the digits 1 or 0 . For each $x_{i}$ there can be an infinite number of digits required, so we write $x_{i}=0 . x_{i 1} x_{i 2} x_{i 3} \ldots$. But this list is supposed to be comprehensive, so we should be able to find any given number in this list. We shall construct a number that cannot possibly be in such a list: let $\bar{x}=0 . \overline{x_{11}} \overline{x_{22}} \overline{x_{33}} \ldots$ where $\overline{x_{n n}}=1-x_{n n}$, i.e. is $\overline{x_{n n}}=0$ when $x_{n n}=1$ and vice versa. Then $\bar{x}$ is connected to every number in the list but is unlike any number in the list in at least one digit, which means it cannot possibly be included in the list. Therefore it cannot be possible to enumerate all such decimal expansions, making $[0,1]$ and therefore also $\mathbb{R}$ uncountably infinite.

### 2.6. FAQ 6: Complex Variables.

FAQ 6a. What justifies Euler's formula $z=r e^{i \theta}$ ? Let $z=x+i y, x, y \in \mathbb{R}, i=\sqrt{-1}$. Plot $z$ on an Argand diagram taking $(1, i)$ as orthogonal basis vectors. Consider the polar coordinates $r, \theta$ of $z$. Change of coordinates gives $x=r \cos \theta, y=r \sin \theta$, where $r=|z|=x^{2}+y^{2}$. Then $z=r(\cos \theta+i \sin \theta)=r$ cis $\theta$. Now, (out of a hat) write the infinite series expansion of $e^{x}=1+x+x^{2} / 2!+x^{3} / 3!+\ldots+x^{n} / n$ !. Formally evaluate at $i x$. And separate terms into the infinite series for $\cos \theta+i \sin \theta$. This completes the proof. Items to note: 1) What gives the Taylor series of the real functions? (FAQ7). 2) What would inspire the idea to consider the exponential? 3) Why should $(1, i)$ be orthogonal units?

FAQ 6b. What motivates complex numbers as rotations? An amateur and self-taught mathematician Argand made observations and delivered proofs $(1806,1813)$ of the highest importance to the budding theory of complex variables in the 19th century when he 1) proposed the Argand diagram for plotting complex numbers in the plane, 2) proposed that multiplication by $i$ be interpreted as a $90^{\circ}$ rotation, and 3 ) proved rigorously that all polynomials split in $\mathbb{C}$ (Fundamental Theorem of Algebra). https://en.wikipedia.org/wiki/Jean-Robert_Argand Not many mathematicians have scooped Gauss (1830). But it took Gauss and Cauchy's eminence to make widespread the theory.

FAQ 6c. What is the most beautiful equation? to the famous formula $-1=e^{i \pi}$, often listed by mathematicians as the most beautiful equation in mathematics (https://en.wikipedia.org/wiki/Euler's_identity). But the attribution to Euler may not be correct. Certainly Euler knew an equivalent fact, Roger Cotes knew of the $e^{i \theta}=$ cis $\theta$ fact, and it is believed that Johann Bernoulli may have known this and conveyed it to Euler (https://books.google.lu/books?id=sohHs7Ex0sYC\&pg=PA4\&redir_esc=y\#v=onepagerq\&f=false (Sandifer, How Euler Did it, 2007, http://eulerarchive.maa.org/hedi/, https://www. amazon.co.uk/How-Euler-Even-More-Spectrum/dp/0883855844/ref=sr_1_1? ie=UTF8\&qid=1466896664 http://eulerarchive.maa.org/hedi/, 76 Columns in How Euler Did It, collected in 2 books by Edward Sandifer, who had a stroke in 2009 and is currently recovering.)

### 2.7. FAQ 7: Analysis.

FAQ 7a. Just like every number is a sequence of rationals, can we say the same for functions?
FAQ 7b. How to take the Taylor Series of a function? Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function, i.e. infinitely differentiable. Suppose $f$ has an infinite series expansion about $x_{0}$ (for simplicity, you could think of $x_{0}=0$ ):

$$
f\left(x-x_{0}\right)=\sum_{k=0}^{\infty} a_{k}\left(x-x_{0}\right)^{k}=a_{0}+a_{1}\left(x-x_{0}\right)+a_{2}\left(x-x_{0}\right)^{2}+\ldots
$$

Then $a_{0}=f\left(x_{0}\right)$. Similarly, $a_{1}=f^{\prime}\left(x_{0}\right)$, and $a_{2}=f^{\prime \prime}\left(x_{0}\right) / 2$ !, etc.
Observe that $f^{(n)}(x)=\sum_{k=n}^{\infty} a_{n} n!\left(x-x_{0}\right)^{k-n}$, so $f\left(x_{0}\right)=a_{n} n$ !, i.e.

$$
a_{n}=f^{(n)}\left(x_{0}\right) / n!
$$

### 2.8. FAQ9: Algebraic and Transcendental Numbers.

FAQ 9a. What do we know about algebraic and transcendental irrational numbers? The first irrationals were discovered by the Pythagoreans, and arise very simply from the incommensurability of one dimensional units and ratios with two dimensional lengths, e.g. the diagonal of a unit square. Then geometry provides a host more. Most roots, certainly roots of all primes, but also all roots that are not perfect squares. Cube roots. Chords of a circle. Arc-length of a circle of unit diameter ( $\pi$ ). Areas provide more: area of a circle. The Greeks and mathematicians for almost 2000 years tried to find a square (with rational side lengths) which would have the equivalent area of a circle (the quadrature problem). This is also not possible, as the required root would be $(\sqrt{\pi})$, which is transcendental.

But demonstrating irrationality is not easy. The sequence of demonstrations began with Euler and $e$.
An outstanding lay summary is https://en.wikipedia.org/wiki/Transcendental_number_theory, and a corresponding technical summary is [36. See Appendix 7.

FAQ 9b. What is the significance of polynomial equations in defining algebraic irrational numbers? If a number is the root of a polynomial with rational coefficients, then it is motivated by the solution to an algebraically constructed equation, one which has a meaningful constructive existence. We accept, for example, that $\sqrt{2}$ is meaningful, because it is the diagonal of a unit square. It is also the root
of $x^{2}-2=0$, which should have a solution. Similarly then $i$ is meaningful as it is the root of $x^{2}+1=0$. All the complex roots of unit are meaningful as they are roots of $x^{n}-1=0$.

So we classify algebraic numbers by the smallest degree polynomial for which they are a solution. Rational numbers are algebraic of degree 1. $\sqrt{2}$ and $i$ are of degree 2. $\sqrt[3]{2}$ is of degree 3 . $\sqrt[n]{2}$ is of degree $n$.

For polynomials up to degree 4 , we can further say that such algebraic numbers are themselves constructible using algebraic operations and surds (the taking of roots), as we have solution formulas expressible in radicals up to quartic. But the Galois theory shows this is not the case for quintic and higher polynomials. So all algebraic numbers of degree 4 or less are expressible by radicals. Some algebraic numbers of degree 5 and higher may be expressible by radicals, e.g. $\sqrt[n]{2}$, but there will be polynomials whose solutions while they exist and are algebraic, will not able able to be expressed in closed form using algebraic operations and radicals. (FAQ 9c.)

FAQ 9c. What can we say about roots of polynomials of higher degree? How can we express them if there is no general closed form for them? We know they exist by the Fundamental Theorem of Algebra. They are algebraic irrationals because they are the solution to finite degree polynomial with rational coefficients.

But what are they like, and how do we construct them, calculate with them, approximate them (decimals, fractions, continued fractions) if we cannot write a closed form solution equation, which Galois theory establishes is not possible in general for every polynomial of degree 5 and higher?

See http://math.stackexchange.com/questions/657168/understanding-non-solvable-algebraic-numbers and the answer by Jack Schmidt (University of Kentucky) (http://www.ms.uky.edu/~jack/)

The key idea is Kroneckar's construction (using the irreducible polynomial to which they are a solution), or companion matrix, i.e. creating a matrix represention of the number field extension, in the same way that one can do so for complex numbers using $2 \times 2$ matrices. This means an algebraic number of degree $n$ has a representation in the set of $n \times n$ matrices chosen from a subfield of $M_{n}(\mathbb{R})$.

This ties into the notion that matrices generalize numbers.
FAQ 9d. What are known classes of irrationals and why are they irrational? All of the following numbers are known to be irrational. The digits of their decimal expansions are infinite without having any infinitely repeating sub-sequence; they are all computable because each is the limit of rational sequences that converge with every additional digit.

- From geometry:
(1) $\sqrt{2}$ (diagonal of a unit square),
(2) $\sqrt[3]{2}$ (duplication of a unit cube),
(3) $\pi$ (ratio of circumference to diameter),
(4) $\sqrt{\pi}$ (side length of square with area equal to a unit circle).
(5) $\sqrt[n]{p} \quad \forall n \in \mathbb{N}$ for every prime $p$, and we know from Euclid that there are infinitely many of these.
- From analysis:
(1) $e$ (growth rate under continuous compounding)..$^{71}$
- From logic:
(1) binary decimal expansions whose digits encode a parameterized decision problem, for example setting the nth binary digit to 1 if the nth integer is prime, else 0 .

FAQ 9e. What are the rationals dense in the reals? The rationals are dense in the reals meaning that in the neighborhood of any real point $\alpha$ there is always a rational point. The proof is easy. Write out the decimal expansion of $\alpha=d_{0} \cdot d_{1} d_{2} \ldots$ and for any given $\epsilon$ stop at the first $k$ that gives $1 / 10^{k}<\epsilon$. Then the resulting number $\alpha_{k}$ is rational (finitely many digits), and $\alpha-\epsilon<\alpha_{k}<\alpha<\alpha_{k}+1 / 10^{k}<\alpha+\epsilon$, which establishes that there are rationals are within any $\epsilon$-neighborhood of $\alpha$.

It turns out the the irrationals are as also dense in the real, i.e. around every real point there is also an irrational. (Show this.)

FAQ 9f. Why are the majority of square roots irrational? We give the proofs that all of the following are irrational: $\sqrt{2}, \sqrt{3}, \sqrt{u}$ for prime $u$, and $\sqrt{n}$ for $n$ not a perfect square. We also give the generalization for all higher roots. The lemmas used are collected in Appendix 9.
$\sqrt{2}$ is irrational. Proof Sketch: Proof is by contradiction. Suppose not, i.e. $x=\sqrt{2} \in \mathbb{Q}$. Then we can express $x$ as a fraction in lowest terms, i.e. $\exists p, q \in \mathbb{N}, q>0$ such that $x=p / q$ with $(p, q)=1$. (The last
statement is the lowest terms requirement, i.e. $p, q$ are coprime, aka relatively prime, which simply means that the greatest common divisor (gcd) is 1 or in other words that $p, q$ share no common factors apart from 1.) Now:

$$
x^{2}=(p / q)^{2}=p^{2} / q^{2}=2 \Rightarrow p^{2}=2 q^{2}
$$

First equality is by substitution, second by multiplicative property of exponents and negative exponents as reciprocals (Lemma 3), third is by definition of root 2 . Then:

$$
\begin{equation*}
p^{2}=2 q^{2} \Rightarrow 2 \mid \text { RHS } \Rightarrow 2\left|p^{2}=p p \Rightarrow 2\right| p(*) \Rightarrow 4\left|p^{2} \Rightarrow 2\right| q^{2} \Rightarrow 2 \mid q(*) \Rightarrow(p, q) \geq 2 \tag{1}
\end{equation*}
$$

The middle steps $\left(^{*}\right.$ ) are true because only two even numbers can be an even product (two odds give an odd). The last statement is that $p, q$ share 2 as a common divisor which contradicts the assumption that $x$ could be expressed as a rational number in lowest terms. So $x=\sqrt{2}$ must be irrational.
$\sqrt{3}$ is irrational. Proof Sketch: Proof follows the same lines replacing 2 with 3 in (1):

$$
p^{2}=3 q^{2} \Rightarrow 3\left|3 q^{2} \Rightarrow 3\right| p^{2}=p p \Rightarrow 3|p(*) \Rightarrow 9| p^{2} \Rightarrow 9\left|3 q^{2} \Rightarrow 3\right| q^{2} \Rightarrow 3 \mid q(*) \Rightarrow(p, q) \geq 3
$$

The middle steps $\left(^{*}\right)$ are now true because a prime divisor of a product must divide one of the two factors (Lemma 1). The conclusion again contradicts the assumption that if the root were rational, it would be able to be expressed as a fraction in lowest terms. So $x=\sqrt{3}$ must be irrational.
$\sqrt{u}$ is irrational for any prime $u$. Proof Sketch: Proof is identical with $u$ replacing 3. The argument follows exactly because $u$ is prime, and for prime $u$ if $u \mid a b$ then either $u \mid a$ or $u \mid b$ (Lemma 1)

All are irrational: $\sqrt[3]{2}, \sqrt[3]{3}, \sqrt[3]{u}, \sqrt[m]{2}, \sqrt[m]{3}, \sqrt[m]{u}$ for any prime $u$ and any root $m \in \mathbb{N}^{+}$. Proof Sketch: The proof follows identical lines as above.
$\sqrt{n}, \sqrt[m]{n}$ are irrational for all $n$ not a perfect square, cube, etc. Proof Sketch: $(\sqrt{n})$ If $n$ is a perfect square, then $\sqrt{n} \in$ naturals. Otherwise, $n$ has a unique prime factorization (UPF) with at least one prime $b$ having an odd power (Lemma), so $n=a b^{q}$, with $q$ odd and ( $a, b^{q}$ ) =1 (coprime). We have $\sqrt{a b}=\sqrt{a} \sqrt{b}$ (Lemma) so $\sqrt{n}=c \sqrt{b}$, and since $b$ is prime, $\sqrt{b}$ is irrational and the unique such term (all others are different or are $b$ (even powers)), and product with no other number can yield a rational number except with a multiple of $\sqrt{b}$ - this is essentially the fact that $\sqrt{b}$ is in a field extension that includes $\sqrt{b}$ as an independent element, i.e. not able to be formed by linear combination of any other numbers. (This proof outline needs improvement.)

The majority of higher roots are irrational: $\sqrt[m]{2}, \sqrt[3]{u}$ for any prime $u, \sqrt[m]{2}, \sqrt[m]{u}, \sqrt[m]{n}$ for $n$ not (? a perfect square or n-cube). Need to compare sizes of countable infinity. Are they all measure 0 ? (See FAQ 5e)

FAQ 10. How are vectors a generalization of number? A vector is a tuple (ordered set) of numbers. The word 'tuple' is itself an extraction of the root of triple, quadruple, quintuple, sextuple, octuple, $n$-tuple. ${ }^{72}$ A vector is like a number because of the vector space operations defined for it: scalar multiplication (stretching), addition. Like a number, a vector has length, and the standard Euclidean norm means that length scales linearly. So $\|c v\|=c\|v\|$ for vectors $v$ and scalars $c$. A vector is a point in a space, and a point is like a number, so essentially, it is the correspondence between numbers and points, and the algebraic structure defined for collections of points cause vectors to behave in the same way as numbers, i.e. one can treat the vector as a single coherent object in many cases, disregarding whether it is one- or many-dimensional.

## Appendix A. Foundations

Edges of Logic: The halting problem asks for a general algorithm that can determine, for any given program and input, whether the program will eventually halt. The problem has been shown by Alan Turing to be undecidable over a Turing machine, which means, such a machine cannot determine in finite time whether any arbitrary program and input will eventually halt.

An undecidable statement is one which it is impossible to determine whether it is always true or always false.

Metalogical concepts: Sound means every provable sentence is valid. Consistent means it is not possible to prove a contradiction from the axioms of the theory. So a consistent theory is sound. Complete means every grammatically valid sentence (so the sentence or its negation) is provable. Effective means there is a proof-checking algorithm that can decide whether any given proof is correct or not. Decidable means there is an algorithm which is capable of deciding the validity of every statement in the language.

Propositional Logic. The objects are propositions. The connectives are AND, OR, NOT, from which one can obtain IF and IFF. Propositional Logic is sound, complete, effective, and decidable.

First order logic (predicate calculus) introduced by C.S. Pierce and Frege. The objects are propositions and variables. Variables can range over elements in a set (specified using predicates), but not over sets (higher order logic). The connectives are propositional connectives and predicates FOR-ALL and THEREEXISTS. For example: for all numbers $\epsilon>0$, etc. ${ }^{73}$. FOL is a complete logical system: sound, complete, and effective, but not decidable. ${ }^{74}$ Set theory (ZFC) can be formulated in FOL, as can number theory (Peano arithmetic). But notice that analysis cannot, as it relies on quantification over sets, and indeed any notion of infinity cannot, even the natural numbers as a set. So FOL is fine for computing, but not for discussions of infinity. ${ }^{75}$

Godel's Completeness Theorem (1929) showed that there are sound, complete, and effective systems for first-order logic. But

Godel's Incompleteness Theorems (there are two) showed the limitations for formal systems. They show that effective first order theories that include arithmetic (both addition and multiplication) cannot be both consistent and complete. For example, Peano Arithmetic and Second Order Arithmetic are both incomplete. This means that there are statements in number theory which may be true but for which we can never obtain a proof either way within the given axiom system. ${ }^{76}$ "Gdel's theorem shows that, in theories that include a small portion of number theory, a complete and consistent finite list of axioms can never be created: each time a new statement is added as an axiom, there are other true statements that still cannot be proved, even with the new axiom. If an axiom is ever added that makes the system complete, it does so at the cost of making the system inconsistent." (2nd URL GIT) Note that ZFC is incomplete as an axiom system - there is a statement which is neither provable or disprovable from ZFC. This means there are statements in set theory for which we may also not have a proof.

Second and higher order logics. These allow variables to be predicated over sets. E.g. for all subsets $A \subset \mathbb{N}$, etc. As such, SOL allows predicating over relations or functions. Why do we need SOL? Because it is not possible to formulate the least upper bound property of the real numbers without it, since this is a statement that asserts something is true for all bounded subsets of real numbers. ${ }^{77}$ Why is SOL problematic? Because allowing predicates to range over all sets, means that Godel's incompleteness theorem holds: there is no deductive system (no notion of provability) for all SOL formulas that ensures simultaneously soundness, completeness, and effectiveness.

ZFC: Zermelo-Fraenkel-Choice is an axiomatization of set theory, formalizable in first order logic, that avoids Russell's paradox. ZFC has been shown to be independent of the continuum hypothesis (CH), and of the independence of Axiom of Choice from the other ZF axioms (sort of like the parallel postulate in geoemtry). The truth value of these statements can be decided by adding axioms to ZFC, e.g. large cardinal axioms (see Feferman). A criticism of ZFC is that it is much stronger than is needed to do most of mathematics - Peano Arithmetic and Second Order Arithmetic are weaker systems which suffice. ${ }^{78}$ The problem with ZFC is that while it can be formulated in FOL, it is not complete.

Peano arithmetic: is a first-order arithmetic system formalizes the natural numbers and their arithmetic properties as a mathematical group.

Second order arithmetic: is an axiomatic system that is stronger than Peano arithemtic, but weaker than ZFC, and yet is strong enough to prove most of classical mathematics. It allows quantification over sets, so allows real analysis.

Reverse mathematics uses second order arithmetic to investigate the axiomatic foundations of mathematics itself. ${ }^{79}$

## Appendix B. Bibliography

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## Appendix C. Babylonian Mathematics

## Appendix D. Arithmetic

In arithmetic, we learn what was known to mathematicians through to the 16 th century, and what was taught to the merchants and user community of mathematics. We have the Treviso Arithetic of [1478], Robert Recorde's English arithmetic from [1540s], and Rafael Bombelli's comprehensive L'Algebra of [1572].

The history of arithmetic has a few surprises.
Subtraction has never been standardized (DE Smith). There are several effective subtraction algorithms, many quite old (DE Smith). The modern standard method of borrowing with the crutch of cancelling numbers became widespread in textbooks in the US post 1937, though it appeared earlier in UK, and was apparently taught in any case in the schools.

The fastest method is the method of equal additions, in which one adds a power of 10 to both numbers, but to one place value in one instance, and to the next place up as a unit. (Ross)

The earliest arithmetics were Lilavati by Bhaskara (1150) Craft of Nombrynge (1300s) by John of Hollywood (Sacrobosco), Treviso Arithmetic (1478), Robert Record's Groundwork of Arte (1540) and Whetstone of Witte (1557)

1450, is from the Art of Nombryng, a translation of De Arte Numerandi, a 13th-century treatise sometimes attributed to the monk Johannes de Sacrobosco.

Multiplication by negative numbers. Why is $-1 \times-1=1$ ? Because of the distributive law. Consistency of the laws becomes the justification for extending the definition of number so as to eliminate cases that would otherwise be undefined.

## Powers and roots - rules for rational (fractional) exponents.

## The use of the abacus.

Arithmetic texts from early history. European arithmetics. Arithmetics in India. Arithmetics in Arabia - these were not as common due in Arabia following the Greek tradition favoring geometry over arithmetic. Arithmetics in ancient Egypt and Babylonia.

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## Appendix E. Rational Numbers

We want to understand:

- From fractions to decimals
- From decimals to fractions
- Representations of 1 , e.g. $1 / 2+1 / 3+1 / 6=1$ and $1 / 2+1 / 4+1 / 5+1 / 20=1$, and $1 / 2+1 / 4$ $+1 / 8+1 / 16+\ldots=1$
- Egyptian fractions, e.g. $2 / 3=1 / 2+1 / 6$ and $3 / 4=1 / 2+1 / 4$
- General representations, e.g. $1 / 6=1 / 2-1 / 3$ and $1 / 3=1 / 2-1 / 6$
- Partial fractions
- Fractional sizes (fractional tree)
- Continued fractions
- Representation of algebraic fractions

Let us collect the evidence. (Exercise: calculate by hand a decimal table of small rational fractions.)

## Rational fractions as decimals.

(1) $1 / 2=0.5$
(2) $1 / 3=0 . \overline{3}, 3 x=9$
(3) $1 / 4=0.25$
(4) $1 / 5=0.2$
(5) $1 / 6=0.1 \overline{6}, 1 / 6=1 / 10+6 / 90$
(6) $1 / 7=0 . \overline{142857}, 142,857 x=999,999$
(7) $1 / 8=0.125$
(8) $1 / 9=0 . \overline{1}$, key for all identical digit expansions, $1 x=9$
(9) $1 / 10=0.1$
(10) $1 / 11=0 . \overline{09}, 9 x=99$
(11) $1 / 12=0.08 \overline{3}, 1 / 12=8 / 100+3 / 900$
(12) $1 / 13=0 . \overline{076923}, 76,923 x=999,999=709,230+230,769=10 \times 76,923+3 \times 76,923$ (notice the rotation of digits when multiplying by 3 )
(13) $1 / 14=0.0 \overline{714285}, 714,285 x=9,999,990=7,142,850+2,857,140$ (again notice the rotation of digits when multiplying by 4)
(14) $1 / 15=0.0 \overline{6}, 6 x=90$
(15) $1 / 16=0.0625,1 / 2 \times 1 / 8=(1 / 8) / 2$
(16) $1 / 17=0 . \overline{0588235294117647}, A x=10^{1} 5-1$, (remarkable repeating digits without repeated blocks)
(17) $1 / 18=0.0 \overline{5}$
(18) $1 / 19=0 . \overline{052631578947368421}$
(19) $1 / 20=0.05$
(20) $1 / 21=0 . \overline{047619}$
(21) $1 / 28=0.03 \overline{571428}$
(22) $1 / 90=0.0 \overline{1}$
(23) $1 / 99=0 . \overline{01}$
(24) $1 / 900=0.00 \overline{1}$
(25) $1 / 990=0.0 \overline{01}$
(26) $1 / 999=0 . \overline{001}$

## Questions.

(1) Every rational fraction has either a finite expansion or an infinitely repeating block of digits.
(2) What determines whether a rational fraction has a finite decimal expansion or an infinitely repeating block?
(3) What determines the size of a fraction's finite block or infinitely repeating block of digits?
(4) Why are $1 / 3,1 / 5,1 / 11,1 / 13$ different from $1 / 7,1 / 17$ and $1 / 19$ ? The latter have 6,16 and 18 decimals distinct before the repetition happens on the 7 th, 17 th and 19 th places respectively. The former have no correlation between the prime divisor and the number of repeating decimals.
(5) Is there some general fact about cyclic rotation of digits in certain fractions, e.g. $1 / 7,1 / 14,1 / 21$, $1 / 28$.
Let a rational fraction $r=p / q, p \in \mathbb{Z}, q \in \mathbb{N}^{+},(p, q)=1$ (the last condition is that $p, q$ are coprime, aka relatively prime, or in technical language have greatest common divisor (gcd) of 1 , or in plain language share no factors in common). Let the decimal expansion of $r=n . d_{1} d_{2} d_{3} \ldots$

Key observations:
(1) The three types of fraction depends on the relative primality of $q$ with 10 :
(2) for coprime $q$, e.g. $1 / 3,1 / 7,1 / 9,1 / 11,1 / 13,1 / 17,1 / 19,1 / 21$, all distinct digits are infinitely repeating, and the fraction is $d_{1} d_{2} \ldots d_{n} / 10^{n}-1$.
(3) For $q$ that share all factors with 10 , i.e. are divisible by powers of 2 and 5 e.g. $1 / 2,1 / 4,1 / 5,1 / 8$, $1 / 10,1 / 16,1 / 20$, i.e. $q=2^{a} 5^{b}$, the decimal expansion is finite, i.e. must terminate: $d_{1} d_{2} \ldots d_{n} / 10^{n}$.
(4) For all other $q$, i.e. that are divisible by either 2 or 5 and that have a factor coprime with 10, e.g. $1 / 6,1 / 14,1 / 15,1 / 18$, these decimal expansions are more complex, and fall under the general rule.
(5) Notice that $1 / q$ doesn't get small very quickly: $1 / 100=0.01$.

## From Decimals to Fractions.

Proposition 1. Every decimal expansion $d$ has a fractional part [d] having (1) finitely many digits

$$
[d]=0 . d_{1} d_{2} \ldots d_{n}
$$

or (2) infinitely many digits with a finite repeating block, i.e.

$$
[d]=0 . d_{1} d_{2} d_{3} \ldots d_{k-1} \overline{d_{k} \ldots d_{n}}
$$

or (3) infinitely many digits with no finite repeating block, i.e.

$$
[d]=0 . d_{1} d_{2} \ldots
$$

Cases 1 and 2 are rational fractions, Case 3 is an irrational fraction. Case 1:

$$
[d]=d_{1} d_{2} \ldots d_{n} / 10^{n}
$$

Case 2:

$$
[d]=\frac{d_{1} d_{2} \ldots d_{k-1}}{10^{k-1}}+\frac{d_{k} \ldots d_{n}}{\left(10^{n-k+1}-1\right) \times 10^{k-1}} .
$$

Exercises:
(1) 0.5
(2) 0.25
(3) $0 . \overline{3} \mathrm{~A}: 1 / 3$
(4) $0 . \overline{1}$ A: $1 / 9$
(5) $0 . \overline{2} \mathrm{~A}: 2 / 9$
(6) $0 . \overline{6} \mathrm{~A}: 6 / 9=2 / 3$
(7) $0 . \overline{5}$ A: $5 / 9$
(8) $0 . \overline{51}$ A: $51 / 99$
(9) $0 . \overline{504}$ A: $504 / 999$
(10) $0 . \overline{12345}$ A: $12345 / 99999$
(11) 0.12345 A: $12345 / 10^{5}$
(12) $0.12345 \overline{505} \mathrm{~A}: 12345 / 10^{5}+504 /\left(999 \times 10^{5}\right)$

References ${ }^{80}$

## Appendix F. Irrational Numbers

Irrational numbers are not rational, i.e. whose fractional part is somehow not expressible as the ratio of two lengths (Exercise). They are of two types: algebraic- or transcendental irrationals. An algebraic number is the root (solution) of an algebraic (polynomial) equation with rational coefficients (integer if denominators are cleared): $a_{0} x^{n}+a_{1} x^{n-1}+\ldots+a_{n-1} x+a_{n}=0$. (If the polynomial is monic, i.e. $a_{0}=1$, then we call this an algebraic integer - even though the number may not be an integer at all. Exercise) All other numbers are transcendental.

Here's what we know [22]:
(1) We know that there are only countably many algebraic irrationals, but that the reals are uncountable (Cantor's diagonal argument), therefore the transcendental irrationals must have uncountable cardinality.
(2) Despite these being the major part of the real numbers, it is surprisingly difficult to prove that a given number is transcendental and also to construct examples of transcendental numbers.
(3) Algebraic irrational numbers and algebraic integers are closed, the former is a field, the latter is a ring. (Exercise)
(4) $\pi$ and $2^{\sqrt[3]{2}}$ are transcendental. There are others: http://sprott.physics.wisc.edu/pickover/trans.html

## Appendix G. Functions

$2+3,2-3,2 \times 3,2 / 3,2^{3}$ - these are the five arithmetic operations on numbers, defined pragmatically when both operands are integers, and extended to all numbers (including octonions). Multiplication is repeated addition, powering repeated multiplication. Division is repeated subtraction. Note, power and division have connections: $2^{-3}=1 / 2^{3}$, i.e. division is multiplication by inverse, which is defined for all numbers in an algebraic field.

What are root and log? They are inverse operations on the power function: given the number $y=2^{3}$, $\sqrt[3]{y}$ extracts the root $2 ; \log _{2} y=3$ extracts the power 3 .

## Appendix H. Project Guide

## Number.

(1) Early History of Numbers and Counting. Read Gullberg (birth of numbers) 25]. Read Ifrah (history of numbers) [28]. Read beliefs about numeracy in cultures 42]. Read about the Piraha peoples, Chomsky's views on language, and Dan Everett's counterviews. [23]
(2) Large Numbers. Read Archimedes (Sand Reckoner) 3]
(3) Higher infinities. Read Dedekind (Theory of Numbers) 12
(4) Construction of the Reals. Read Feferman (Number System) [17. Read Landau [34]
(5) Set Theory axioms. Read Feferman's papers [16, [19, [20, [21], 18]
(6) Algebraic and Transcendental Numbers. Read Gelfond [22].
(7) Elliptic Numbers (arc-length of an ellipse).
(8) Are algebraic integers ever integers? Do they have to be? ${ }^{81}$
(9) Show that $\sqrt{2}, \sqrt{p}$, with $p$ a prime, $\sqrt[n]{2}$, and $\sqrt[n]{2}$ are algebraic irrationals.
(10) How to approximate a general irrational using fractions? How about the specific ones above?
(11) Show that both rationals and irrationals are dense in $\mathbb{R}$, i.e. that each of their points is a boundary point with their complement.
(12) Unlock the meaning / secret / construction of continued fractions.
(13) Why does $\sqrt{2}$ have a very suggestive continued fraction representation? 30]

## Science.

(1) How Mathematics is like science. Read Courant [8]

## Time.

(1) Read Galileo's Dream (fiction), by Kim Stanley Robinson.
(2) Read Newton's Principia for a statement on time.

## Motion.

(1) Read the Pre-Socratic philosophers (Freeman), Meno (Plato), and Aristotle for early views on motion.

## Space.

(1) Read Minkowski on Spacetime.

## Early Mathematics.

(1) Ishango bone.
(2) Babylonian mathematics (Robson)
(3) Greek mathematics and especially their concept of number (separate from length), Diophantus (Bourbaki) and Archimedes method of exhaustion.

## Algebraic Systems.

(1) Are the integers a group? ring? field? Are the complex numbers? Are the Gaussian integers, i.e. $a+b i \in \mathbb{C}$ with $a, b \in \mathbb{Z} ?^{82}$
(2) Are algebraic irrationals a field? a ring? a group? ${ }^{83}$
(3) Are algebraic integers (solutions of monic polynomial) a ring? ${ }^{84}$
(4) Are the irrationals a ring? ${ }^{85}$ A group? ${ }^{86}$

## Lemmas for proofs that majority of roots are irrational (FAQ9).

Lemma 1. $u|a b \Rightarrow u| a$ OR $u \mid b$ if $u$ is prime.. Proof Sketch: Every integer has a unique prime factorization (UPF). If $u$ is not $\operatorname{UPF}(a)$ or $\operatorname{UPF}(b)$ then the only way $u$ can be in $\operatorname{UPF}(a b)$ is if $u=r s$ with $r$ in $\operatorname{UPF}(a)$ and $s$ in $\operatorname{UPF}(b)$, but $u$ is prime so has no factors other than 1 and itself so this cannot be the case, hence either $u \mid a$ or $u \mid b$.

Lemma 2. $\sqrt{a b}=\sqrt{a} \sqrt{b}$. Proof Sketch:

$$
(\sqrt{a b})^{2}=a b=(\sqrt{a})^{2}(\sqrt{b})^{2}=(\sqrt{a} \sqrt{b})^{2} .
$$

First and second equalities are by definition of square root, third equality is by property of exponents (Lemma 3).

To prove now the general statement that the majority of roots are irrational, we need lemmas about roots and exponents and unique prime factorization of integers.

Lemma 3a. Exponents are multiplicative: $(a b)^{2}=a^{2} b^{2}$. Proof Sketch:

$$
(a b)^{2}=(a b)(a b)=a(b a) b=a(a b) b=(a a)(b b)=a^{2} b^{2} .
$$

First and final equality are by definition of square, second and fourth by associativity of multiplication in integers, third by commutativity of multiplication.

Lemma 3b. Exponents are additive: $a^{b} a^{c}=a^{b+c}$.
Lemma 3c. Negative exponents are reciprocals: $a^{-1}=1 / a$.
Show that the irrationality of roots proof fails for $\sqrt{4}$ or $\sqrt{n}$ when $n$ is a perfect square. The failure comes in (1):

$$
p^{2}=4 q^{2} \Rightarrow 4\left|p^{2} \sim \Rightarrow 4\right| p
$$

because 4 is not prime, so two cases: either $4 \mid p$ or $2 \mid p$. The former would lead to a contradiction, but the latter does not. Indeed, it is the case $(p=2, q=1)$. The same occurs with $\sqrt{n}$. We cannot make the crucial leap from $n \mid p^{2}$ to $n \mid p$ unless $n$ since $n=r r$, i.e. is itself a perfect square.


[^0]:    E-mail address: assad.ebrahim@mathscitech.org.
    Date: June 19, 2016.
    *Clay tokens in Neolithic settlements in the Middle East and Eastern Europe have been discovered dating in an unbroken period from 8,000 B.C.E. to 3,000 B.C.E. with the development of cuneiform writing on clay tablets in Babylon. [26, p. $33 . \mathrm{ff}$, [?] Counting tokens, abacus, knots in quipo, tally sticks, beans in gourds, were all used to maintain commercial records and execute transactions. While there are claims of counting, tallying, and calendar tracking based on artefacts dating to 35,000 B.C.E. (Lebombo bone) ${ }^{8}$ and to conjectured knowledge of the primes going back to 22,000 B.C.E. (Ishango bone), ${ }^{9}$ in both these cases the evidence is far from certain. 31 ${ }^{10}$
    ${ }^{\dagger}$ the challenge is to keep it aligned with the seasons, the secret of which is found in the regular movement of the stars. ${ }^{19}$

[^1]:    *algebraic, root of polynomial $x^{2}-x-1=0$.

[^2]:    ${ }^{10}$ Notches may have been created to improve grip when used as cutting or scraping tools (the Ishango bone) or as a weapon (the wolf bone), with the marks inscribed in groups of four to match where fingers curl about the haft of the tool. It is an ever-present danger in creating mathematical fictions where none exist in the desire to find reflections of modernity in ancient or prehistoric records.

[^3]:    *http://www.math.tamu.edu/~dallen/history/babylon/babylon.html
    http://www.historyworld.net/wrldhis/PlainTextHistories.asp?historyid=ab57
    ${ }^{\dagger}$ http://www.dartmouth.edu/~toxmetal/toxic-metals/more-metals/copper-history.html
    http://www.iceman.it/en/axe http://www.iceman.it/en/copperage
    $\sqrt[8]{\text { http://www.anvilfire.com/21centbs/stories/rsmith/mesopotamia_1.htm }}$
    4 http://factsanddetails.com/world/cat56/sub363/item2216.html

[^4]:    http://it.stlawu.edu/~dmelvill/mesomath/history.html

