Catalysts in the Growth of Mathematics

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Abstract

The growth of mathematics has had many encouraging forces: societal, technological, cultural. These have served to accelerate mathematics and have been accelerated in turn, in many cases the pair becoming locked into a mutually beneficial resonance that has dramatically energized both.

In this article, I look at some of the significant catalysts, from the rise of the leisured class in ancient times to the impact of computing in modern times.

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The First Catalyst: Necessity, Advancement, Leisure The initial catalysts for the development of mathematics (and science with it) were necessity, the desire for supremacy, and the growth of leisure.

Necessity was driven by the need to count, tax, administer, divide, plant, and harvest – the use of mathematical knowledge for matters of subsistence and governance.

> "According to most accounts geometry was first discovered among the Egyptians and originated in the measuring of their lands. This was necessary for them because the Nile overflows and obliterates the boundaries between their properties." - Proclus ([Bur], p.35 (Greek, 410-485AD))

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The desire for advancement consisted of the desire to be able to maintain a group of individuals successfully against all odds, including the ability to repulse attackers. This required planning and preparation for an unknown or partially knowable future, and hence gave rise to developments in astronomy for weather prediction, astrology for the understanding of portents, and to the need for logistics of maintaining armies, imposing and collecting tribute, and ruling subjugated tribes and cities.

The growth of leisure due to the advance of a settled society was likewise instrumental. A leisure class devoted to the study of knowledge is associated with tremendous advances in each of the ancient civilizations:

> "The mathematical sciences originated in the neighborhood of Egypt, because there the priestly class was allowed leisure." - Aristotle, Metaphysics.

It was among the leisure class that the moves toward abstraction and theoretical considerations were made.

> "Although the initial emphasis was on utilitarian mathematics, the subject began eventually to be studied for its own sake. Algebra evolved ultimately from the techniques of calculation, and theoretical geometry began with land measurement." - [Bur] p.35

The First Concept that Exploded Mathematics: Abstract Number The first concept that exploded mathematics was the abstraction of the number concept from the specific object(s) that it described. So $2+3=5$ abstracted from whether the discussion is about sheep, beads or people. With the development of this concept came simple arithmetic, familiarity with arithmetic laws, the representation of numbers through number symbols, spoken and written, computations, and finally the development of algebra through the symbolic use of numbers. All of these developments characterized the mathematics of the prehistoric through the early civilizations: Babylonian, Egyptian, Chinese, Mayan, Indian.[1]

Ease of Calculation, Number Bases, and the Positional Number System Early on the Babylonians found that a positional system with a base makes calculation easier. They used base 60. Later on, due to the influence of the Hindus and Arabs, a base 10 system became prominent. This base 10 system has been standardized throughout much of the world due to the leadership of the French in the formation of the International System of Measurement which set down the metric standards.

Surveying – The Flood Plains of the Nile The need for the Egyptian courts to regulate the land management and taxation systems in the Nile river delta lead to advances in geometry applied to the problem of performing yearly

[1] See [Ebr05], [Ebr06], [Lit49].

surveys of land in the floodplains, after the swollen Nile had washed away all markers and had physically altered the formation of the land.

Astronomy – The Astronomical Observations of the Babylonians The need for precise astronomical information to guide the ceremonial and agricultural activities of the Mesopotamian and Babylonian civilizations lead to ever increasing accuracy in astronomical observations and related calculations.

An Explosive Catalyst: Incommensurable (Irrational) Numbers The existence of irrational numbers (incommensurable measures) led the Greeks to reject number (and the arguments of the Zeno school to address the difficulties by embracing the infinite and the infinity) and place their faith instead in geometric constructibility (which was able to represent such measures that defied the Greek concept of number: $\pi, \sqrt{2}$, etc.). The problem was that geometric constructibility is itself not a good basis for all number, but this fact was not discovered until the sword-point of Galois theory killed this notion. Once this happened in the late 1800s, what was left in its stead was hitherto undesirable paradoxical approach of Zeno (convergent series) and the method of exhaustion of Archimedes. These provided the seeds for the modern approach to the infinite: analysis, limits, etc.[2]

The Second Concept that Exploded Mathematics: Axiomatic Arrangement The second concept that exploded mathematics was the axiomatic arrangement of material. This idea was introduced by the ancient Greeks, led by Thales and his student Pythagoras in the organization of geometric truths. Though many of these truths were known to the Babylonians and Egyptians prior to the Greeks, their arrangement of their knowledge was not axiomatic. The axiomatic arrangement excited the thorough exploration of geometry in the Greek civilization, and its orderly arrangement culminating in the Elements of Euclid.[3]

Mechanical Statics – The Application of Geometry to Mechanical Statics Bridge building, aqueducts, pyramics, all monuments of man are testimony to an understanding of geometry and mechnical statics.

Mercantilism and Arithmetic Algorithms The rise of mercantilism during the early renaissance in Europe lead to the perfection and propagation of the arithmetic arts.

A Liberating Catalyst: the improvement of mathematical notation and the rise of Symbolical Algebra A notation that allows efficient use, a notation that suggests connections and organization of manipulative procedures that are simple and give the correct answer are great spurs to advance of

[2] See [Ebr09a], [Ebr09b].

[3] See [Ebr08], [Pan], [Ale56].

knowledge, for labor places a horizon on the ambitions of most men and thus labor-saving devices are the womb from within which great advances issue.

The value of good notation to a mathematician cannot be stressed enough. Inasmuch as mathematics is a discipline in which concepts are replaced by symbols, it is of the utmost importance that these symbols contain about them the necessary reminders of those aspects of the concepts that should always be borne in mind. Thus, the appendages in the form of superscripts, subscripts, and other decorations are not meant to drive one to distraction, but rather to provide the facile user of the mathematical method the means to retain in the forefront of her mind those details that are important to bear in mind, appearing there as she inscribes her symbols. Good notation frees the mind, can bring out patterns, banishes the need for words.

Quantitative and Experimental Science – The Attempt to Measure and Describe Through Experiment and Observation Bacon and Galileo are examples of the new mindset in Europe that gradually brought European thought out of the grip of religious Scholasticism and prepared the way for the Scientific Revolution and Renaissance. Both men were firm empiricists, and sought to measure, describe, and analyze through observation and experiment.

An Accelerating Catalyst: The Introduction of Coordinates to Algebraize Geometry The merging of algebra and geometry was a development that gathered separate streams in mathematics into a river that began, then, to gather pace toward the explosion into the Calculus. Important figures were Fermat and Descartes.

The Third Concept that Exploded Mathematics: The Calculus The third concept that exploded mathematics was the formalization of the notion of change and how it accumulates. This idea was anticipated in the ancient Greeks, in the work of Archimedes and Appollonius. It was presaged by work by Galileo, Fermat and by Wallis. It was brought to light in the work of Newton and Leibniz in the ideas of the calculus.

This concept of change was a critical new idea in mathematics and it led to widespread application of the methods of the calculus to many diverse areas of science.

The Penetration of Physics – the Calculus, Differential Equations, and Fourier Analysis The post-Calculus era was one in which mathematicians were physicists, scientists, experimentalists and science was a unified whole. Mathematics was one tool in both the description and the discovery of solution methods in experimental science. We have Faraday and Maxwell, experimenter and theoretician. We have Brahe and Kepler, experimenter and theoretician. We have many others.

The Fourth Concept that Exploded Mathematics: Fourier Analysis The fourth concept that exploded mathematics was the decomposition of arbitrary functions into a trigonometric series. This idea was expounded and demonstrated physically by Fourier in the study of heat. It brought new methods for solving differential and partial differential equations, opening up vast additional areas in the study of science. It also accelerated the development of a vast area of mathematical analysis to deal with the theoretical framework in which the Fourier analysis could be established with sufficient rigor. [4]

The Fifth Concept that Exploded Mathematics: Set Theory The fifth concept that exploded mathematics was the language of set theory and its ability to penetrate, through reasoning, areas beyond physical investigation (e.g. cardinalities of the infinite, the number sets, the continuum). This new language allowed the expression of abstract ideas independent of a physical model to constrain the discussion. Now the discussion of abstractions and generality could proceed without having to live within a metaphor (e.g. the geometrical metaphor for abstract linear algebra, topology, or functional analysis, also abstract algebra). The metaphor is certainly there, and it influences the mathematics constantly, but the expression of content can be done independently of a metaphor. The language of set theory unleashed the pure mathematics by providing purer tools for its exposition, making the exposition, organization, and presentation of mathematics much closer to the Greek ideal of pure deduction.

The Continued Development of Abstract Systems (Models) The development of abstract systems to serve as models for other systems or for applications, has allowed mathematics to be simplified, extended, and made dramatically more powerful and applicable. The extensions, however, have lead to the increase in complexity.[5] Examples of these models are: linear algebra, functional analysis.

Of course, commensurate with this increasing ability to deal with concepts abstractly and the increasing desire to do all things abstractly, has come increasing distance between the commonplace understandings of lay persons and the specialized language, methods, modes of thought and accepted forms of expression of professional mathematicians. This brings with it greater tendencies for isolation, and for the splintering of efforts among the scientific community, as mathematics begins to assert a mode that is not by necessity the mode of development of the physical sciences.

Statistics Statistics arose from the study of data from experiment: the astronomical observations of Tycho Brahe that lead to Kepler's revolutionary three postulates of planetary astronomy, and Gauss's astronomical observations that led him to develop the method of the least squares treatment of data.

[4] See [Hub98].

[5] See [Ped89].

Optimization Scarcity in all its forms makes optimization both necessary and valuable. Competition is a form of scarcity of consumers, suppliers, commodities, etc. The shortage of time and the expansiveness of desire and ambition makes efficiency (a form of optimization) valuable. All of these increase the value of studies that improve efficiency, the efficient allocation of resources, the saving of time, effort, or material, and the ability to outcompete one's competitors. These increase the value of mathematics, since it is able to assist in many of these areas.

The role of physics The explosion in physics with the discovery of electricity and magnetism, special and general theories of relativity, quantum physics, atomic and nuclear physics, and particle physics, has been a major client and muse for much mathematics, and still continues to be with its string and gauge theories of particle physics.

The role of the computer The rise of the electronic computer and the steady exponential increase in its capacity (Moore's Law) and the corresponding advances in software technology (compilers, programming languages, applications such as Octave/Matlab, R/S, Maxima/Mathematics/Maple, GeoGebra/Geometer's Sketchpad, etc.), have opened up a realm of lightning fast, inexpensive, precise and virtually unlimited computational power, and made numerical computations, analysis, simulation, modeling, and the approximate solution of large scale problems of all kinds, possible. This, in and of itself, has allowed the reach of mathematics to be extended into areas that would have been considered intractable before. This, singlehandedly, has given rise to the flourishing new fields of numerical methods, scientific computing, numerical modeling and simulation, the development of algorithms, and discrete mathematics.

The rise of electronic mathematical platforms and software to assist in the performance of mathematical calculations has been of tremendous help in the flowering of mathematical ideas and the expansion of its application.

Nowadays mathematics is everywhere, and its use is routine in a vast number of diverse areas. Certainly, mathematics undergirds all of the technology of the modern world, from satellites to computerized functions in modern cars, to modern manufacturing and quality control, computer generated graphics for the cinema, and sensors in our dishwashers, security alarms, the route optimization of the Internet and Google search, not to mention all of the military hardware and software. And what has made this possible is the immeasurable savings of labor in the performance of the quadrillions of calculations that are made electronically, automatically, and continuously by trillions of tiny electronic chips.

The role of engineering In the investigation of the physical and mechanical field phenomena of the continuum (heat, flow, diffusion, convection, elasticity, strength of materials, etc.), major advances have been achieved in the mathematical and physical understanding, and in the computational modeling and simulation methods. All of these have led to corresponding progress in engineering accomplishment, which has in turn stoked engineering ambition, leading to the need to investigate further, hence fueling further progress in mathematics, physics, and numerical computation. From Aerospace, Oceanography, and Materials Science, the applications of these new studies has been profound. As examples of the progress just mentioned, are the Fourier theory, partial differential equations, finite element methods, fluid mechanics and dynamics, turbulent flows, etc.

The deepening of pure mathematics

Liberation from Classical problems and Classical Thought The discovery of the non-Euclidean geometries liberated mathematics from physical axioms, and moved it toward a study of mathematical structures.

The discovery by Galois of the impossibility of the three classical Greek challenges (doubling the cube, quadrature of the circle, and trisection of an angle), as well as the solution by radicals of all polynomials, was a dramatic testimony to the fact that there exists a mathematics of structure that is indeed much deeper than the common elements that classical mathematics and the mathematics of the 1800s had percived.

This was further reinforced by the development of the real numbers, and the discovery that the infinite cardinality of the integers, rationals, and algebraic irrational numbers are the same, but that there are higher orders of infinity that exist, and indeed it must be accepted that the transcendental irrational numbers are of this higher order of infinity, and of the same cardinality as that of the continuum.

The algebraic discoveries of Lagrange, Abel, and Galois, led to the study of permutations, groups, fields, field extensions, and uncovered structure in algebra that was considered to be more fundamental that the structure of number, that had been viewed as fundamental in the periods prior.

The shift of pure mathematics towards abstract structures Pure mathematics is now concerned with general structures, and views possibly disparate application fields as fields whose mathematics has a similar, pure mathematical, under-structure. This is seen in abstract algebra, topology, functional analysis, and many other fields of modern mathemtics. This shift to abstract algebraic structures was further reinforced by the discovery of more complex algebraic structures in the mathematics required to work with the growing advance of new application areas in physics, etc. Vectors, matrices, non-commutative particles in physics, complex numbers, quaternions, all of these were part of the menagerie of the mathematics of the 1800s.

Establishing of analytic and set theoretic foundations Meanwhile, the set theory and the invention of the real numbers liberated Analysis, and bolstered the discovery of Fourier and his infinite series method of approximating arbitrary functions. The notion of arbitrary functions however, coupled with the vast higher order of infinity of both the real numbers and set theory, began to uncover monster after monster in analysis, requiring a slew of new definitions, boundaries, and characteristics to fence in the "good" and distinguish them from the "wild" functions. But many of the "wild" functions were being found to be useful in the new landscape of probability (and measure theory), distribution theory (Dirac-δ function, etc.).

And indeed, the core notions of calculus (differentiability and integrability), were being desired for the new wilder functions, thus leading to more sophisticated machinery to allow that to happen (Frechet derivative, Lipschitz continuity, almost entire and almost everywhere notions, measure and integration of Lesbesgue and others, and the subdifferentiability and non-continuous calculus of modern optimization).

The flowering of geometry The proliferation of Geometries, too, made the subject ready for the discoveries of Riemann and Klein that there are general ways to characterize most geometries – through the formulation of a metric tensor (Riemann), and through an understanding of continuous groups and what they hold invariant (Klein). Indeed, the proliferation of geometries of arbitrary curvature and structure and the desire to bring analytical methods to these as well as look at deformations of surfaces, led to the study of topology, homotopy, and the theory of manifolds and algebraic topology.

Telecommunications and Information technology The most abstract pursuits in number theory and abstract algebra have found applications in cryptography, encryption, error correcting codes, and bar code scanning, etc., all of which have transformed industrial operations, telecommunications, and information technology.

The interplay between empirical science and pure mathematics This deepening of pure mathematics has thus been validated by the experiences of those working in the areas that were undergoing dramatic expansion in the applications of mathematics and in the phenomena to which mathematics was being applied. For they were experiencing things that the pure mathematicians were able to make precise with their new structures and methods. This has led to the present unspoken faith that the unfettered pursuit of pure mathematics is a thing of value, justified by the many instances since the 1800s in which abstract mathematics has uncovered discoveries that have been found to be of great value in the precise formulation of these areas.

Further Reading The following provide additional reading on the material of this article: [AKL63], [Boya], [Boyb], [Bur], [Cro], [Ebr09b], [Ebr09a], [Gal94], [Hoy], [Kle86], [Kli], [Lit49], [Pan], [Wal06].

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