

# History of the formulas and algorithms for $\pi$

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## Abstract

Throughout more than two millennia many formulas have been obtained, some of them beautiful, to calculate the number  $\pi$ . Among them, we can find series, infinite products, expansions as continued fractions and expansions using radicals. Some expressions which are (amazingly) related to  $\pi$  have been evaluated. In addition, a continual battle has been waged just to break the records computing digits of this number; records have been set using rapidly converging series, ultra fast algorithms and really surprising ones, calculating isolated digits. The development of powerful computers has played a fundamental role in these achievements of calculus.

## 1 First formula: Archimedes' algorithm

For a period of approximately 1800 years, Archimedes' algorithm, proved around 250 BC, was the most efficient way to calculate  $\pi$ . The idea consisted of considering a circle with unit diameter with regular circumscribed and inscribed polygons of  $3 \cdot 2^n$  sides. Let  $a_n$  and  $b_n$  denote their perimeters, respectively. He proved the following relations using only geometrical reasoning:

$$a_1 = 2\sqrt{3}, \quad b_1 = 3, \quad a_{n+1} = \frac{2a_n b_n}{a_n + b_n}, \quad b_{n+1} = \sqrt{a_{n+1} b_n}. \quad (1)$$

Obviously,  $b_n < \pi < a_n$  and both sequences  $a_n$  and  $b_n$  converge to  $\pi$ . It is an algorithm which nowadays can be easily proved using elementary trigonometry. Indeed, if  $k_n = 3 \cdot 2^n$ , we can write

$$a_n = k_n \tan \frac{\pi}{k_n}, \quad b_n = k_n \sin \frac{\pi}{k_n}.$$

Then

$$\frac{2a_n b_n}{a_n + b_n} = 2k_n \frac{\tan \frac{\pi}{k_n} \sin \frac{\pi}{k_n}}{\tan \frac{\pi}{k_n} + \sin \frac{\pi}{k_n}} = 2k_n \tan \frac{\pi}{2k_n} = a_{n+1}.$$

On the other hand,

$$\sqrt{a_{n+1} b_n} = \sqrt{2k_n \tan \frac{\pi}{2k_n} \cdot k_n \sin \frac{\pi}{k_n}} = 2k_n \sqrt{\tan \frac{\pi}{2k_n} \sin \frac{\pi}{2k_n} \cos \frac{\pi}{2k_n}} = b_{n+1}.$$

Convergence in this algorithm is linear; after 5 steps we get 3 digits of  $\pi$ . To see this, we use the inequalities  $\tan x < x$  and  $1 - \cos x < \frac{x^2}{2}$  to get

$$a_n - b_n = k_n \tan \frac{\pi}{k_n} \left(1 - \cos \frac{\pi}{k_n}\right) < \frac{\pi^3}{2k_n^2} = \frac{\pi^3}{18 \cdot 2^{2n}} < \frac{\pi^3}{18 \cdot 1000^{n/5}}.$$

With  $n = 7$ , this algorithm gives the estimate  $3.1415 < \pi < 3.1417$ .

It is enlightening to point out that during the age of Archimedes neither decimal notation nor any other positional notation was known, so he stated the result he obtained in terms of polygons with 96 sides ( $n = 5$ ) using fractions:

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}.$$

This approximation determines two-place accuracy; this means  $\pi \approx 3.14$ .

## 2 No great news until 1800 years later

For a period of approximately 1800 years all formulas that were obtained for  $\pi$  were based on the idea of approximating a circle using regular polygons, and they took the shape of iterative algorithms. In 1593 Viète obtained a different expression by means of a pretty expansion with infinitely many nested square roots:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots \quad (2)$$

As with all the previous methods, Viète's is based on geometry and consists of considering a circle whose diameter is 2 and approximating its area, that is,  $\pi$ , by the area of an inscribed square, the area of an inscribed octagon, the area of an inscribed 16-sided polygon, and so on. In this way Viète found  $\pi$  accurately to nine places: 3.141592653.

Around 1650, analytical methods of Descartes and Fermat, who were forerunners of the development of Calculus which would be carried out soon, started replacing geometrical methods. One of the formulas for  $\pi$  at this stage is the famous infinite product due to Wallis (1655):

$$\frac{2}{\pi} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{7}{8} \cdot \frac{9}{8} \cdots \quad (3)$$

In 1999, Osler proved a formula which includes as particular cases both Viète's formula and Wallis' infinite product. We will see this in section 10.

## 3 The early stages of the Calculus: New Formulas for $\pi$

Between 1665 and 1680, Newton and Leibniz independently developed the first basic notions of Calculus, which supplied us with a powerful tool not only for mathematics but also for physics and other sciences. These methods, as it could not be otherwise, also had repercussions in obtaining new formulas for  $\pi$ . For example, one obtained by Newton is

$$\pi = \frac{3\sqrt{3}}{4} + 2 - \frac{3}{4} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(n+1)(2n+5)16^n}, \quad (4)$$

by which he approximated  $\pi$  to 15 places in 1665. Newton reached this formula by noticing that the area which is bounded by a circle with the equation

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

and the straight lines  $x = \frac{1}{4}$  and  $y = 0$  is equal to the difference between the area of a circular sector that takes up an angle of  $60^\circ$  and its corresponding triangle. Therefore

$$\int_0^{1/4} y \, dx = \int_0^{1/4} \sqrt{x - x^2} \, dx = \int_0^{1/4} x^{1/2}(1 - x)^{1/2} \, dx = \frac{\pi}{24} - \frac{\sqrt{3}}{32}.$$

Newton's formula is obtained from here by expanding  $(1 - x)^{1/2}$  using the binomial formula, which he had proved, and then integrating term by term. Formulas such as (4) are nowadays known as binomial sums.

A series similar to Newton's one but simpler is

$$\frac{\pi}{3} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{(2n+1)16^n}, \quad (5)$$

which we get by substituting  $x = \frac{1}{4}$  into the series expansion (see [6], p. 386)

$$\frac{\arcsin 2x}{2x} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2n+1} x^{2n}.$$

Another interesting example of a binomial sum is the following obtained by Comtet in 1974:

$$\pi^4 = \frac{3240}{17} \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}} \quad (6)$$

which can be deduced from the identity [8]

$$\sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}} = 2 \int_0^{\pi/3} \theta \log^2 \left( 2 \sin \frac{\theta}{2} \right) d\theta.$$

In 1671 Gregory integrating the following geometric series term by term:

$$1 - x^2 + x^4 - x^6 + \dots = \frac{1}{1 + x^2}, \quad |x| < 1,$$

got the formula (which has his name)

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \quad |x| \leq 1,$$

which would become known as the power series expansion of the function  $\arctan x$ . Substituting  $x = 1$ , Leibniz got in 1674 the series for  $\pi$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \quad (7)$$

It is strange that Gregory did not do it himself, probably because he was not interested in slowly convergent series. Sharp put  $x = \sqrt{3}/3$  in Gregory's formula and obtained the series

$$\frac{\pi}{6} = \frac{\sqrt{3}}{3} \left( 1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \dots \right) \quad (8)$$

which allowed him to achieve the record of 71 digits for  $\pi$  in 1699. It is surprising that Gregory's formula and Sharp's particular case have been found in the Sanskrit texts that the Indian learned person Nilakantha Somayaji had written 200 years before.

In 1665 Brouncker got the pretty expansion as a continued fraction

$$\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \frac{9^2}{2 + \dots}}}}} \quad (9)$$

About 100 years later, Euler proved it in a clever way. He starts by proving the equality

$$a_1 + a_1 a_2 + a_1 a_2 a_3 + a_1 a_2 a_3 a_4 \dots = \frac{a_1}{1 - \frac{a_2}{1 + a_2 - \frac{a_3}{1 + a_3 - \frac{a_4}{1 + a_4 - \dots}}}}$$

which motivates him to rewrite Gregory's series as

$$\arctan x = x + x \left( \frac{-x^2}{3} \right) + x \left( \frac{-x^2}{3} \right) \left( \frac{-3x^2}{5} \right) + x \left( \frac{-x^2}{3} \right) \left( \frac{-3x^2}{5} \right) \left( \frac{-5x^2}{7} \right) + \dots$$

and then he gets Brouncker's formula by substituting  $x = 1$  and using the continued fraction.

A recent example of expanding  $\pi$  as a continued fraction is

$$\pi = 3 + \frac{1^2}{6 + \frac{3^2}{6 + \frac{5^2}{6 + \frac{7^2}{6 + \frac{9^2}{6 + \dots}}}}} \quad (10)$$

obtained by Lange [21] in 1999.

## 4 Machin type series

In 1706, John Machin proved the following formula (which is known today as Machin's formula):

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}. \quad (11)$$

Using Gregory's formula he rewrote it as a series and in this way he got the first 100 digits of  $\pi$ . The following identities are similar, but they lead to more slowly convergent series:

$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}, \quad \frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}. \quad (12)$$

The proof of a known Machin type series expression consists of an ordinary checking of an identity between elementary operations with complex numbers. We use

$$\frac{(5+i)^4}{239+i} = 2+2i$$

to prove Machin's formula; it is enough to make the arguments equal.

Gauss also took part in the search for rapidly convergent series for  $\pi$  and he proved the following equality in 1863:

$$\frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}. \quad (13)$$

It is like the preceding ones, but it involves one more term.

In 2002 Kanada and his team, using a certain type of trick to optimize computer calculations, got a record computing  $\pi$  by means of the series

$$\frac{\pi}{4} = 12 \arctan \frac{1}{49} + 32 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} + 12 \arctan \frac{1}{110443}, \quad (14)$$

$$\frac{\pi}{4} = 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12943}. \quad (15)$$

The first one was obtained by Takano in 1982 (he is a high school teacher and a song composer). The second one was obtained by Störmer in 1896. The record consisted of more than a billion of digits of  $\pi$ .

## 5 Formulas by the brilliant Swiss mathematician L. Euler

Great mathematicians, Leibniz and the Bernoulli brothers Jacob (1654-1705) and Johann (1667-1748) among them, had attempted to evaluate (with no success) the infinite sum

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \quad (16)$$

Finally, Euler proved in 1736 that its value is  $\pi^2/6$ . To do it he started with the series

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

and the equation

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots = 0.$$

The solutions of this equation should be the solutions of  $\sin x = 0$ , except  $x = 0$ , that is:  $\pm\pi$ ,  $\pm 2\pi$ ,  $\pm 3\pi$ ,  $\dots$ . He imagined the infinite series as a polynomial and he broke it down into factors

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots = \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \dots$$

He finally got the proof by setting the  $x^2$  coefficients equal. He also proved many similar formulas. Among them we can find

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32} \quad (17)$$

and the not-so-similar

$$\sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}{n^3} = \frac{\pi^4}{72}. \quad (18)$$

Nowadays formulas like the last one are known as Euler sums [14]. The following one has an interesting history

$$\sum_{n=1}^{\infty} \frac{\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)^2}{n^2} = \frac{17\pi^4}{360}, \quad (19)$$

because it was observed numerically by the student Au Yeung. J. Borwein, his professor, was skeptical of the result but after checking it with greater precision he got to prove it. Now, we know that the formula had been proved long before by H. Sandham [29]. However, the rediscovery of this forgotten formula has had an important effect because it has motivated the organized study of these kind of formulas.

While studying the divergence of the series of reciprocals of the prime numbers, Euler considered geometric progressions:

$$1 + \frac{1}{p^2} + \frac{1}{p^4} + \frac{1}{p^6} + \frac{1}{p^8} + \dots = \frac{1}{1 - \frac{1}{p^2}},$$

and he observed that by multiplying over all primes  $p$  we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \prod_p \left(1 - \frac{1}{p^2}\right)^{-1}, \quad (20)$$

which links the number  $\pi$  with prime numbers.

## 6 An outstanding forgotten algorithm: C. F. Gauss' Formula

When he was only 14 years old, Gauss (one of the greatest mathematicians through the ages) considered the following iterative process:

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}$$

and he defined the arithmetic-geometric mean of  $a$  and  $b$  as the common limit

$$M(a, b) = \lim a_n = \lim b_n.$$

In 1800, his investigations concerning the arithmetic-geometric mean lead him to the wonderful formula (see [1], p. 94-102)

$$\sum_{k=0}^{\infty} 2^k (a_k^2 - b_k^2) = 1 - \frac{2M^2}{\pi}, \quad (21)$$

where  $a_0 = 1$ ,  $b_0 = 1/\sqrt{2}$  and  $M$  is the arithmetic-geometric mean of  $a_0$  and  $b_0$ .

For no clear reasons, Gauss did not make use of his formula to calculate  $\pi$ . In 1970, it was rediscovered independently by Eugene Salamin and Richard Brent [25], [9] who expressed it appropriately for iterative calculation:

$$\lim \frac{2a_n^2}{1 - \sum_{k=0}^n 2^k (a_k^2 - b_k^2)} = \pi.$$

It can be proved that the convergence to  $\pi$  is quadratic, that is, from  $n$  to  $n + 1$  the number of correct digits is doubled. Brent and Salamin programmed this algorithm on a computer. It let them reach in 1976 a record approximating  $\pi$  to 3 million digits, while the record until that moment had been 1 million digits, achieved using Machin-type formulas.

## 7 Series for $1/\pi$ from an incredible mathematical genius: S. Ramanujan

Most of the Indian mathematical genius Ramanujan's works are related with modular function theory, which originated from studying elliptic integrals of the first and second kind

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 t}} dt, \quad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t} dt.$$

In 1914 these investigations lead him to obtain extraordinary series for  $1/\pi$  completely different from those known until that moment. These new series were of the following type:

$$\sum_{n=0}^{\infty} B_n z^n (bn + a) = \frac{1}{\pi},$$

where  $b, a > 0$  and  $-1 < z < 1$  are algebraic numbers. They were classified into 4 families according to the following possibilities for  $B_n$ :

$$\frac{(2n)!^3}{n!^6}, \quad \frac{(4n)!}{n!^4}, \quad \frac{(2n)!(3n)!}{n!^5}, \quad \frac{(6n)!}{(3n)!n!^3}.$$

He partly proved formulas which allow us to obtain the first family of series [24] and he gave 3 examples, two of which are

$$\sum_{n=0}^{\infty} \frac{(2n)!^3}{n!^6} \frac{1}{28^n} (6n+1) = \frac{4}{\pi}, \quad \sum_{n=0}^{\infty} \frac{(2n)!^3}{n!^6} \frac{1}{2^{12n}} (42n+5) = \frac{16}{\pi}. \quad (22)$$

He said that there exist alternative theories (but he did not detail any of them) which allow us to obtain series from the other 3 families and he gave 14 examples [24] among which we can find the following:

$$\sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1}{396^{4n}} (26390n+1103) = \frac{9801\sqrt{2}}{4\pi}, \quad (23)$$

$$\sum_{n=0}^{\infty} \frac{(2n)!(3n)!}{n!^5} \frac{1}{1458^n} (15n+2) = \frac{27}{4\pi}, \quad (24)$$

$$\sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!n!^3} \frac{1}{54000^n} (11n+1) = \frac{5\sqrt{15}}{6\pi}. \quad (25)$$

Series (23) is incredibly fast, since each term contributes 8 digits of pi. In 1985, W. Gosper programmed this formula on a computer, before its proof was known, and he got a record of 17 million digits for  $\pi$ .

The Borwein brothers, Jonathan and Peter, were the first to prove (in 1985) formulas which allow us to obtain the 4 families of Ramanujan formulas [6] and they showed the identity (23). The Chudnovsky brothers, David and Gregory, developed in [12] a theory to prove the following amazing series formula, which was not discovered by Ramanujan:

$$\sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!n!^3} \frac{(-1)^n}{640320^{3n}} (545140134n+13591409) = \frac{\sqrt{640320^3}}{12\pi}. \quad (26)$$

Each term contributes 14 digits of  $\pi$  and was used to beat records of the calculation of  $\pi$  in 1989, 1991 and 1994, the last one with 4044 million digits. Finally, H. H. Chan, W. C. Liaw and V. Tan found identities [10] which allowed them to prove the following formula, which also had not been discovered by Ramanujan:

$$\sum_{n=0}^{\infty} \frac{(3n)!(2n)!}{n!^5} \frac{(-1)^n}{300^{3n}} (14151n+827) = \frac{1500\sqrt{3}}{\pi}. \quad (27)$$

Very recently B. Berndt and N. Baruah in [3] and [4] followed more closely the initial ideas of Ramanujan as presented in Section 13 of the celebrated paper [24], and proved many new series formulas of this kind with algebraic values of its parameters.



A proof of the families of formulas for  $1/\pi$  can consist in finding functions  $z(q)$ ,  $b(q)$  and  $a(q)$ , related to the Jacobi elliptic theta functions

$$\theta_2(q) = \sum_{n=-\infty}^{n=\infty} q^{(n+1/2)^2}, \quad \theta_3(q) = \sum_{n=-\infty}^{n=\infty} q^{n^2}, \quad \theta_4(q) = \sum_{n=-\infty}^{n=\infty} (-1)^n q^{n^2},$$

such that

$$\sum_{n=0}^{\infty} B_n z^n(q) \{b(q)n + a(q)\} = \frac{1}{\pi}.$$

Evaluating  $z(q)$ ,  $b(q)$  and  $a(q)$  at  $q = \pm e^{-\pi\sqrt{N}}$ , where  $N$  is a rational number, one obtains algebraic numbers and so Ramanujan series.

The Chudnovsky brothers proved that his formula (26) corresponds to  $N = 163$  and has the fastest convergence rate among all those Ramanujan series having a rational value of the parameter  $z$ . They arrived at this conclusion by noticing that 163 is the largest class number [12]. An excellent survey on Ramanujan's series is [5].

## 8 The Borwein brothers' quartic algorithm

The fourth root of the elliptic lambda function and the elliptic alpha function [6], that is,

$$s(q) = \sqrt[4]{\lambda(q)} = \frac{\theta_2(q)}{\theta_3(q)}, \quad \alpha(q) = \frac{1}{\pi} \cdot \frac{1}{\theta_3^4(q)} \left[ 1 + (4q \ln q) \frac{\theta_4'(q)}{\theta_4(q)} \right]$$

satisfy the following wonderful hard-to-prove properties:

$$s(q^4) = \frac{1 - \sqrt[4]{1 - s^4(q)}}{1 + \sqrt[4]{1 - s^4(q)}},$$

$$\alpha(q^4) = \alpha(q)[1 + s(q^4)]^4 + \frac{4 \ln q}{\pi} s(q^4)[1 + s(q^4) + s^2(q^4)],$$

which connect the function values at  $q$  with the values at  $q^4$ , so they allow us to obtain an amazing Borwein family of quartic algorithms [7]. For example, if we let

$$q = e^{-2\pi \cdot 4^n}, \quad s_n = s(q), \quad t_n = \alpha(q),$$

then the values of  $s_0$  and  $t_0$  are algebraic, and we get the following quartic algorithm:

$$s_0 = \sqrt{2} - 1, \quad t_0 = 6 - 4\sqrt{2},$$

$$s_{n+1} = \frac{1 - \sqrt[4]{1 - s_n^4}}{1 + \sqrt[4]{1 - s_n^4}}, \tag{28}$$

$$t_{n+1} = t_n(1 + s_{n+1})^4 - 2^{2n+3} s_{n+1}(1 + s_{n+1} + s_{n+1}^2),$$

where  $t_n$  tends to  $1/\pi$ , and the number of accurate digits is quadrupled in each iteration. In the same way, if we let

$$q = e^{-\pi \cdot 4^n}, \quad s_n = s(q), \quad t_n = \alpha(q),$$

we obtain a quartic algorithm with initial values  $s_0 = 1/\sqrt[4]{2}$  and  $t_0 = 1/2$ . A simple proof of this one is given in [20] using only Gauss' formula (21) and elementary algebra. However, although this proof is much simpler than the Borweins', their method is more general. In addition, J. Borwein and P. Borwein have obtained other quadratic, cubic, etc algorithms. However, it seems that the quartic algorithms are the most efficient.

We might think that  $\pi$  calculation records would always be broken using quadratic, quartic, etc. algorithms. However, that is not true, because such algorithms do not auto-correct their errors and they require doing all the operations to the final desired accuracy. Other ways have been found to improve the accuracy of calculation of series without affecting the final result, allowing them to beat records even though they have the drawback of being only linear.

## 9 The jumbled calculation of $\pi$ digits

In 1995 Peter Borwein and Simon Plouffe realized that the series

$$\ln 2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

could be used to compute one of the binary digits of  $\ln 2$  without calculating the previous digits, by means of the following ingenious formula, where  $\{x\}$  means the fractional part of  $x$ :

$$\{2^k \ln 2\} = \left\{ \left\{ \sum_{n=0}^k \frac{2^{k-n}}{n} \right\} + \sum_{n=k+1}^{\infty} \frac{1}{n2^{n-k}} \right\} = \left\{ \left\{ \sum_{n=0}^k \frac{2^{k-n \bmod n}}{n} \right\} + \sum_{n=k+1}^{\infty} \frac{1}{n2^{n-k}} \right\}.$$

It lets us efficiently obtain the binary digit of  $\ln 2$  in position  $k + 1$ . The main remark to carry out all the involved calculation is that the numerator of the first sum, that is,  $2^{k-n} \bmod n$ , can be computed very quickly. For example, the calculation of  $2^{65}$  does not require 65 multiplications but only 7 if we do it as  $(((((2^2)^2)^2)^2)^2) \cdot 2$  and a very efficient algorithm is used to obtain for example  $2^{65} \bmod 7$ . It consists of reducing modulo 7 after each multiplication instead of doing them in a row.

In 1989 the Borwein brothers claimed that obtaining just one binary digit of  $\pi$  could not be easier than computing all the previous digits. However, after this discovery for  $\ln 2$ , it was not so clear and the team consisting of D. Bailey, P. Borwein and S. Plouffe began to search, using the PSLQ algorithm, for a series for  $\pi$  that would allow them to calculate isolatedly its base 2 digits (see [13], chapter 8).

The PSLQ algorithm is a numerical algorithm which operates to a prearranged accuracy. It links an output of  $n$  integers  $a_1, a_2, \dots, a_n$ , where  $a_i \neq 0$  for any  $i$ , with an input of  $n$  real numbers  $x_1, x_2, \dots, x_n$ , and

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

is verified.

Success arrived taking

$$x_j = \sum_{n=0}^{\infty} \frac{1}{(8n+j)16^n}, \quad j = 1, 2, \dots, 7,$$

and trying to find, using PSLQ, a linear relation with integer coefficients using as input  $\pi$ ,  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$ ,  $x_5$ ,  $x_6$ ,  $x_7$ . The output obtained was this sequence of integers:  $-1, 4, 0, 0, -2, -1, -1, 0$ . That is, they found a series [2] which they called BBP type (BBP are the initials of the team members' names)

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left( \frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) \quad (29)$$

that lets us calculate  $\pi$ 's hexadecimal digits (hence also its binary ones) in isolation. Later, a proof of this formula was obtained from the easy result

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8} dx = \int_0^{1/\sqrt{2}} \sum_{n=0}^{\infty} x^{k-1+8n} dx = \frac{1}{2^{k/2}} \sum_{n=0}^{\infty} \frac{1}{16^n(8n+k)}.$$

Namely, we have

$$\sum_{n=0}^{\infty} \frac{1}{16^n} \left( \frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8} dx,$$

and this last integral can be easily evaluated by substituting  $y = x\sqrt{2}$ , as follows:

$$16 \int_0^1 \frac{4 - 2y^3 - y^4 - y^5}{16 - y^8} dy = \int_0^1 \frac{16y - 16}{(y^2 - 2)(y^2 - 2y + 2)} dy = \pi.$$

As soon as they announced the amazing formula (29), other investigators wanted to find other beautiful BBP type formulas. The easiest one for  $\pi$  was found and proved by Viktor Adamchik and Stan Wagon (see [28] pp. 66 and 67)

$$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \left( \frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3} \right). \quad (30)$$

It can be proved similarly to the previous one, just taking into account the identity

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1+x^4} dx = \int_0^{1/\sqrt{2}} \sum_{n=0}^{\infty} (-1)^n x^{k-1+4n} dx = \frac{1}{2^{k/2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(4n+k)}.$$

In 1997, Fabrice Bellard broke a calculation record using the formula

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(2n+1)} - \frac{1}{64} \sum_{n=0}^{\infty} \frac{(-1)^n}{1024^n} \left( \frac{32}{4n+1} + \frac{8}{4n+2} + \frac{1}{4n+3} \right). \quad (31)$$

The latest record belongs to C. Percival, who has obtained that the 10000 billionth binary digit of  $\pi$  is 0.

There are also known BBP formulas for other values related to  $\pi$ , for example,

$$\pi\sqrt{3} = \frac{9}{32} \sum_{n=0}^{\infty} \frac{1}{64^n} \left( \frac{16}{6n+1} + \frac{8}{6n+2} - \frac{2}{6n+4} - \frac{1}{6n+5} \right) \quad (32)$$

and

$$\pi^2 = \frac{9}{8} \sum_{n=0}^{\infty} \frac{1}{64^n} \left( \frac{16}{(6n+1)^2} - \frac{24}{(6n+2)^2} - \frac{8}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{1}{(6n+5)^2} \right). \quad (33)$$



Then he substituted  $n - 1$  for  $n$ , rewrote the logarithm as  $\ln(k + 2) - \ln(k + 1)$  and the sum on  $k$  as the difference of two sums. He replaced  $k$  with  $k - 1$  in the first of these two sums. Then he applied the binomial relation  $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$  and by exponentiating the expression he got the result.

## 11 Very recent results: Series for $1/\pi$ , $1/\pi^2$ and $1/\pi^3$

T. Sato, in a 2002 conference in Japan, showed the series

$$\sum_{n=0}^{\infty} u_n \left( \frac{\sqrt{5}-1}{2} \right)^{12n} (20n + 10 - 3\sqrt{5}) = \frac{20\sqrt{3} + 9\sqrt{15}}{6\pi}, \quad (37)$$

where  $u_n$  are the Apéry numbers, defined by

$$u_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Soon afterwards this type of series was found for other types of numbers [11], [27]. H. H. Chan and Y. Yang independently developed a theory explaining this kind of formula, which also includes as a particular case Ramanujan type series, and they named them Ramanujan-Sato type formulas.

A radically different explanation for some Ramanujan type formulas, one that does not use the theory of modular functions, is based on the WZ method, due to Herb Wilf and Doron Zeilberger [23]. It lets one automatically prove identities of the shape

$$\sum_{n=0}^{\infty} G(n, k) = \text{Constant}$$

in case the function  $G(n, k)$  is hypergeometric in both of its variables, that is, if the quotients  $G(n + 1, k)/G(n, k)$  and  $G(n, k + 1)/G(n, k)$  are rational functions of  $n$  and  $k$ , respectively. The program EKHAD, written by D. Zeilberger, does the hard work of finding a companion function  $F(n, k)$  to  $G(n, k)$  such that  $F(0, k) = 0$ , and which makes  $F(n, k)$ ,  $G(n, k)$  a Wilf and Zeilberger (WZ) pair, that is, a pair characterized by the property

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k).$$

In addition, Zeilberger has proved that if we define  $H(n, k) = F(n + 1, n + k) + G(n, n + k)$ , then we will obtain an identity associated to the first one:

$$\sum_{n=0}^{\infty} H(n, k) = \sum_{n=0}^{\infty} G(n, k) = \text{Constant}.$$

It is important to emphasize that the WZ method is a symbolic method and not a numerical one like the PSLQ algorithm (see section 9), so the proofs obtained are completely rigorous.

The cost to pay is the difficulty of finding interesting functions that are a component of a WZ pair.

Some identities discovered by the author [17], among them

$$\sum_{n=0}^{\infty} \frac{1}{2^{8n} 2^{4k}} \frac{\binom{2n}{n}^3 \binom{2k}{k}^2}{\binom{n+k}{k}^2} (6n + 4k + 1) = \frac{4}{\pi}, \quad (38)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{12n} 2^{8k}} \frac{\binom{2n}{n}^5 \binom{2k}{k}^4}{\binom{n+k}{k}^4} (20n^2 + 8n + 1 + 24kn + 8k^2 + 4k) = \frac{8}{\pi^2}, \quad (39)$$

are suitable for having an automatic proof with EKHAD. Just to see how the WZ method works, we explain in detail the proof of the identity (38) based on this method. Given the function

$$G(n, k) = \frac{1}{2^{8n} 2^{4k}} \frac{\binom{2n}{n}^3 \binom{2k}{k}^2}{\binom{n+k}{k}^2} (6n + 4k + 1),$$

EKHAD gives the companion

$$F(n, k) = \frac{1}{2^{8n} 2^{4k}} \frac{\binom{2n}{n}^3 \binom{2k}{k}^2}{\binom{n+k}{k}^2} 8n$$

and together they form a WZ pair, that is, they satisfy

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k).$$

We sum over all  $n \geq 0$ , so that a telescoping cancellation is produced on the first member, and we get

$$\sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} G(n, k + 1).$$

This series converges uniformly, so we can write

$$\lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} \lim_{k \rightarrow \infty} G(n, k) = \lim_{k \rightarrow \infty} G(0, k) = \lim_{k \rightarrow \infty} \frac{\binom{2k}{k}^2}{2^{4k}} (4k + 1) = \frac{4}{\pi}$$

and we obtain the result

$$\sum_{n=0}^{\infty} G(n, k) = \frac{4}{\pi}.$$

And, as explained before, we have the double identity

$$\sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} H(n, k) = \frac{4}{\pi},$$

where  $H(n, k) = F(n + 1, n + k) + G(n, n + k)$ . Substituting  $k = 0$ , we obtain the Ramanujan series (22)

$$\sum_{n=0}^{\infty} \frac{(2n)!^3}{n!^6} \frac{1}{2^{8n}} (6n + 1) = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(2n)!^3}{n!^6} \frac{1}{2^{12n}} (42n + 5) = \frac{4}{\pi}. \quad (40)$$

For the function  $G(n, k)$  in the series (39), EKHAD finds its companion

$$F(n, k) = \frac{(-1)^n}{2^{12n} 2^{8k}} \frac{\binom{2n}{n}^5 \binom{2k}{k}^4}{\binom{n+k}{k}^4} \cdot 8n(2n + 4k + 1),$$

and then we can say that the proof of (39) is hidden inside this pair  $F(n, k)$ ,  $G(n, k)$ . We have the double identity

$$\sum_{n=0}^{\infty} G(n, k) = \sum_{n=0}^{\infty} H(n, k) = \frac{8}{\pi^2}.$$

Substituting  $k = 0$ , we obtain the new formulas [15], [17], [19]

$$\sum_{n=0}^{\infty} \frac{(2n)!^5}{n!^{10}} \frac{(-1)^n}{2^{12n}} (20n^2 + 8n + 1) = \frac{1}{16} \sum_{n=0}^{\infty} \frac{(2n)!^5}{n!^{10}} \frac{(-1)^n}{2^{20n}} (820n^2 + 180n + 13) = \frac{8}{\pi^2}. \quad (41)$$

Using the PSLQ algorithm, the author managed to find more series of this type for  $1/\pi^2$  [16], for example,

$$\sum_{n=0}^{\infty} \frac{(6n)!}{n!^6} \frac{(-1)^n}{2880^{3n}} (5418n^2 + 693n + 29) = \frac{128\sqrt{5}}{\pi^2}, \quad (42)$$

whose proof is still unknown. In [18] and [19] we found the same series as the result of a conjecture.

B. Gourevitch found a series for  $1/\pi^3$  [16] using the PSLQ algorithm

$$\sum_{n=0}^{\infty} \frac{(2n)!^7}{n!^{14}} \frac{1}{2^{20n}} (168n^3 + 76n^2 + 14n + 1) = \frac{32}{\pi^3}. \quad (43)$$

It has not been proved yet.

W. Zudilin proved the following new identities [30] by means of quadratic transformations of the identities (41):

$$\sum_{n=0}^{\infty} w_n \frac{(4n)!}{n!^2 (2n)!} (18n^2 - 10n - 3) \frac{1}{(28^5 5^2)^n} = \frac{10\sqrt{5}}{\pi^2}, \quad (44)$$

$$\sum_{n=0}^{\infty} w_n \frac{(4n)!}{n!^2 (2n)!} (1046529n^2 + 227104n + 16032) \frac{1}{(5^4 41^2)^n} = \frac{5^4 41 \sqrt{41}}{\pi^2}, \quad (45)$$

where the  $w_n$  are numbers defined by

$$w_n = \sum_{k=0}^n \binom{2k}{k}^3 \binom{2n-2k}{n-k} 2^{4(n-k)}.$$

With these last formulas we conclude our historical review.

## Acknowledgements

I thank Professors Javier Cilleruelo and Jonathan Sondow for their helpful suggestions and comments. I thank Luis Sánchez Lajusticia for translating the original Spanish manuscript.

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