Evolution of the Function Concept: A Brief Survey

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Introduction. The evolution of the concept of function goes back 4000 years; 3700 of these consist of anticipations. The idea evolved for close to 300 years in intimate connection with problems in calculus and analysis. (A one-sentence definition of analysis as the study of properties of various classes of functions would not be far off the mark.) In fact, the concept of function is one of the distinguishing features of "modern" as against "classical" mathematics. W. L. Schaaf [24, p. 500] goes a step further:

> The keynote of Western culture is the function concept, a notion not even remotely hinted at by any earlier culture. And the function concept is anything but an extension or elaboration of previous number concepts—it is rather a complete emancipation from such notions.

The evolution of the function concept can be seen as a tug of war between two elements, two mental images: the geometric (expressed in the form of a curve) and the algebraic (expressed as a formula—first finite and later allowing infinitely many terms, the so-called "analytic expression"). (See [7, p. 256].) Subsequently, a third element enters, namely, the "logical" definition of function as a correspondence (with a mental image of an input-output machine). In the wake of this development, the geometric conception of function is gradually abandoned. A new tug of war soon ensues (and is, in one form or another, still with us today) between this novel "logical" ("abstract," "synthetic," "postulational") conception of function and the old "algebraic" ("concrete," "analytic," "constructive") conception.

In this article, we will elaborate these points and try to give the reader a sense of the excitement and the challenge that some of the best mathematicians of all time confronted in trying to come to grips with the basic conception of function that we now accept as commonplace.

1. Precalculus Developments. The notion of function in explicit form did not emerge until the beginning of the 18th century, although implicit manifestations of the concept date back to about 2000 B.C. The main reasons that the function concept did not emerge earlier were:

- lack of algebraic prerequisites—the coming to terms with the continuum of real numbers, and the development of symbolic notation;
- lack of motivation. Why define an abstract notion of function unless one had many examples from which to abstract?

In the course of about two hundred years (ca. 1450–1650), there occurred a number of developments that were fundamental to the rise of the function concept:

- Extension of the concept of number to embrace real and (to some extent) even complex numbers (Bombelli, Stifel, et al.);
- The creation of a symbolic algebra (Viète, Descartes, et al.);
- The study of motion as a central problem of science (Kepler, Galileo, et al.);
- The wedding of algebra and geometry (Fermat, Descartes, et al.).

The 17th century witnessed the emergence of modern mathematized science and the invention of analytic geometry. Both of these developments suggested a dynamic, continuous view of the functional relationship as against the static, discrete view held by the ancients.

In the blending of algebra and geometry, the key elements were the introduction of *variables* and the expression of the relationship between variables by means of *equations*. The latter provided a large number of examples of curves (potential functions) for study and set the final stage for the introduction of the function concept. What was lacking was the identification of the independent and dependent variables in an equation:

> Variables are not functions. The concept of function implies a unidirectional relation between an “independent” and a “dependent” variable. But in the case of variables as they occur in mathematical or physical problems, there need not be such a division of roles. And as long as no special independent role is given to one of the variables involved, the variables are not functions but simply variables [2, p. 348].

See [6], [15], [27] for details.

The calculus developed by Newton and Leibniz had not the form that students see today. In particular, it was not a calculus of *functions*. The principal objects of study in 17th-century calculus were (geometric) curves. (For example, the cycloid was introduced geometrically and studied extensively well before it was given as an equation.) In fact, 17th-century analysis originated as a collection of methods for solving problems about curves, such as finding tangents to curves, areas under curves, lengths of curves, and velocities of points moving along curves. Since the problems that gave rise to the calculus were geometric and kinematic in nature, and since Newton and Leibniz were preoccupied with exploiting the marvelous tool that they had created, time and reflection would be required before the calculus could be recast in algebraic form.

The variables associated with a curve were geometric—abscissas, ordinates, subtangents, subnormals, and the radii of curvature of a curve. In 1692, Leibniz introduced the word “function” (see [25, p. 272]) to designate a geometric object associated with a curve. For example, Leibniz asserted that “a tangent is a function of a curve” [12 p. 85].

Newton’s “method of fluxions” applies to “fluents,” not functions. Newton calls his variables “fluents”—the image (as in Leibniz) is geometric, of a point “flowing” along a curve. Newton’s major contribution to the development of the function concept was his use of power series. These were important for the subsequent development of that concept.

As increased emphasis came to be placed on the formulas and equations relating the functions associated with a curve, attention was focused on the role of the symbols appearing in the formulas and equations and thus on the relations holding among these symbols, independent of the original curve. The correspondence (1694–1698) between Leibniz and Johann Bernoulli traces how the lack of a general term to represent quantities dependent on other quantities in such formulas and equations brought about the use of the term "function" as it appears in Bernoulli's definition of 1718 (see[3, p. 9] and [27, p. 57] for details):

> One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants [23, p. 72].

This was the first formal definition of function, although Bernoulli did not explain what "composed in any manner whatever" meant. See [3], [6], [12], [27] for details of this section.

2. Euler's *Introductio in Analysin Infinitorum*. In the first half of the 18th century, we witness a gradual separation of 17th-century analysis from its geometric origin and background. This process of "degeometrization of analysis" [2, p. 345] saw the replacement of the concept of variable, applied to geometric objects, with the concept of function as an algebraic formula. This trend was embodied in Euler's classic *Introductio in Analysin Infinitorum* of 1748, intended as a survey of the concepts and methods of analysis and analytic geometry needed for a study of the calculus.

Euler's *Introductio* was the first work in which the concept of function plays an explicit and central role. In the preface, Euler claims that mathematical analysis is the general science of variables and their functions. He begins by defining a function as an "analytic expression" (that is, a "formula"):

> A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities [23, p. 72].

Euler does not define the term "analytic expression,"[1] but tries to give it meaning by explaining that admissible "analytic expressions" involve the four algebraic operations, roots, exponentials, logarithms, trigonometric functions, derivatives, and integrals. He classifies functions as being algebraic or transcendental; single-valued or multivalued; and implicit or explicit. The *Introductio* contains one of the earliest treatments of trigonometric functions as numerical ratios (see [13]), as well as the earliest algorithmic treatment of logarithms as exponents. The entire approach is algebraic. Not a single picture or drawing appears (in v. 1).

Expansions of functions in power series play a central role in this treatise. In fact, Euler claims that any function can be expanded in a power series: "If anyone doubts this, this doubt will be removed by the expansion of every function" [3, p. 10].[2] This remark was certainly in keeping with the spirit of mathematics in the 18th century.

Hawkins [10, p. 3] summarizes Euler's contribution to the emergence of function as an important concept:

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[1] This term, which will appear often throughout this paper, was formally defined only in the late 19th century (see sec. 7).

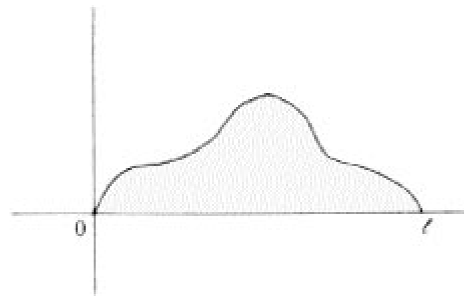
[2] Youschkevitch [27, p. 54] claims that "because of power series the concept of function as analytic expression occupied the central place in mathematical analysis."

> Although the notion of function did not originate with Euler, it was he who first gave it prominence by treating the calculus as a formal theory of functions.

As we shall see, Euler's view of functions was soon to evolve. See [2], [3], [6], [27] for details of the above.

3. The Vibrating-String Controversy. Of crucial importance for the subsequent evolution of the concept of the function was the Vibrating-String Problem:

An elastic string having fixed ends (0 and ℓ, say) is deformed into some initial shape and then released to vibrate. The problem is to determine the function that describes the shape of the string at time *t*.



The controversy centered around the meaning of "function." In fact, Grattan-Guinness suggests that in the controversy over various solutions of this problem, "The whole of eighteenth-century analysis was brought under inspection: the theory of functions, the role of algebra, the real line continuum and the convergence of series . . ." [9, p. 2].

To understand the debates that surrounded the Vibrating-String Problem, we must first mention an "article of faith" of 18th century mathematics:

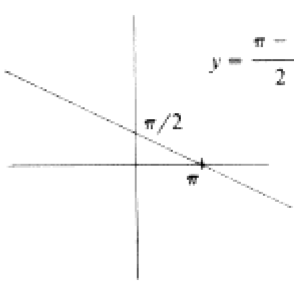
> *If two analytic expressions agree on an interval, they agree everywhere.*

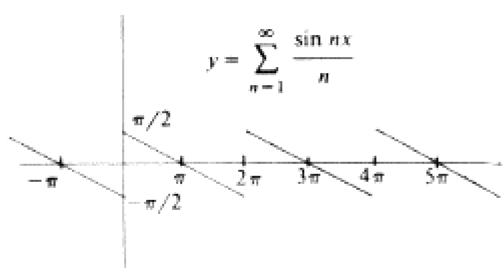
This was not an unnatural assumption, given the type of functions (analytic expressions) considered at that time. On this view, the whole course of a curve given by an analytic expression is determined by any small part of the curve. This implicitly assumes that the independent variable in an analytic expression ranges over the whole domain of real numbers, without restriction.

In view of this, it is baffling (to us) that as early as 1744, Euler wrote to Goldbach stating that

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

(See [27], p. 67.) Here, indeed, is an example of two analytic expressions that agree on the interval $(0, 2\pi)$, but nowhere else:[3]





[3] Euler must surely have recognized this, but "This is not the only occasion on which EULER knew examples which did not comply with his conceptions but which he may have considered to be insignificant exceptions from the general rule" [27, p. 67]. See also [19].

In 1747, d'Alembert solved the Vibrating-String Problem by showing that the motion of the string is governed by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad (a \text{ is a constant}),$$

the so-called *wave equation*. Using the boundary conditions $y(0, t) = 0$ and $y(\ell, t) = 0$, and the initial conditions

$$y(x, 0) = f(x) \quad \text{and} \quad \left.\frac{\partial y}{\partial t}\right|_{t=0} = 0,$$

he solved this partial differential equation to obtain $y(x, t) = [\varphi(x + at) + \varphi(x - at)]/2$ as the "most general" solution of the Vibrating-String Problem, φ being an "arbitrary" function. It follows readily that

$$y(x, 0) = f(x) = \varphi(x) \text{ on } (0, \ell),$$

$$\varphi(x + 2\ell) = \varphi(x),$$

and

$$\varphi(-x) = -\varphi(x).$$

Thus, φ is determined on $(0, \ell)$ by the initial shape of the string, and is continued (by the "article of faith") as an odd periodic function of period 2ℓ.

D'Alembert believed that the function $\varphi(x)$ (and hence $f(x)$) must be an "analytic expression"—that is, it must be given by a formula. (To d'Alembert, these were the only permissible functions.) Moreover, since this analytic expression satisfies the wave equation, it must be twice differentiable.

In 1748, Euler wrote a paper on the same problem in which he agreed completely with d'Alembert concerning the solution but differed from him on its interpretation. Euler contended that d'Alembert's solution was not the "most general," as the latter had claimed. Having himself solved the problem mathematically, Euler claimed his experiments showed that the solution $y(x, t) = [\varphi(x + at) + \varphi(x - at)]/2$ gives the shapes of the string for different values of t, even when the initial shape is not given by a (single) formula. From physical considerations, Euler argued that the initial shape of the string can be given (a) by several analytic expressions in different subintervals of $(0, \ell)$ (say, circular arcs of different radii in different parts of $(0, \ell)$) or, more generally, (b) by a curve drawn free-hand.[4] But according to the "article of faith" prevalent at the time, neither of these two types of initial shapes could be given by a single analytic expression, since such an expression determines the shape of the entire curve by its behavior on any interval, no matter how small. Thus, d'Alembert's solution could not be the most general.

[4] Euler called functions of types (a) and (b) "discontinuous," reserving the word "continuous" for functions given by a single analytic expression. (Thus, he regarded the two branches of a hyperbola as a single continuous function! [18, p.301].) This conception of "continuity" persisted until 1821, when Cauchy gave the definition used nowadays.

D'Alembert, who was much less interested in the vibrations of the string than in the mathematics of the problem, claimed that Euler's argument was "against all rules of analysis." (Euler believed that it is admissible to apply certain of the operations of analysis to arbitrary curves.)[5] Langer [16, p.17] explains the differing views of Euler and d'Alembert concerning the Vibrating-String Problem in terms of their general approach to mathematics:

> Euler's temperament was an imaginative one. He looked for guidance in large measure to practical considerations and physical intuition, and combined with a phenomenal ingenuity, an almost naive faith in the infallibility of mathematical formulas and the results of manipulations upon them. D'Alembert was a more critical mind, much less susceptible to conviction by formalisms. A personality of impeccable scientific integrity, he was never inclined to minimize short-comings that he recognized, be they in his own work or in that of others.

Daniel Bernoulli entered the picture in 1753 by giving yet another solution of the Vibrating-String Problem. Bernoulli, who was essentially a physicist, based his argument on the physics of the problem and the known facts about musical vibrations (discovered earlier by Rameau et al.). It was generally recognized at the time that musical sounds (and, in particular, vibrations of a "musical" string) are composed of fundamental frequencies and their harmonic overtones. This physical evidence, and some "loose" mathematical reasoning, convinced Bernoulli that the solution to the Vibrating-String Problem must be given by

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \cos\frac{n\pi a t}{\ell}.$$

This, of course, meant that an arbitrary function $f(x)$ can be represented on $(0, \ell)$ by a series of sines,

$$y(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}.$$

(Bernoulli was only interested in solving a physical problem, and did not give a definition of function. By an "arbitrary function" he meant an "arbitrary shape" of the vibrating string.)

Both Euler and d'Alembert (as well as other mathematicians of that time) found Bernoulli's solution absurd. Relying on the 18th century "article of faith," they argued that since $f(x)$ and the sine series $\Sigma_{n=1}^{\infty} b_n \sin(n\pi x/\ell)$ agree on $(0, \ell)$, they must agree everywhere. But then one arrived at the manifestly absurd conclusion that an "arbitrary" function $f(x)$ is odd and periodic. (Since Bernoulli's initial shape of the string was given by an analytic expression, Euler rejected Bernoulli's solution as being the most general solution.) Bernoulli retorted that d'Alembert's and Euler's solutions constitute "beautiful mathematics but what has it to do with vibrating strings?" [22, p. 78].

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[5] Euler's, but not d'Alembert's "rules of analysis" would allow him to admit, for example, the curve  as the initial shape of a vibrating string. For, Euler would argue that one could change the shape of the curve at the "top" by an infinitely small amount and thus "smooth" it out. Since infinitesimal changes were ignored in analysis, this would have no effect on the solution.

The debate lasted for several more years (it was joined later by Lagrange) and then died down without being resolved. Ravetz [22, p. 81] characterized the essence of the debate as one between d'Alembert's mathematical world, Bernoulli's physical world, and Euler's "no-man's land" between the two. The debate did, however, have important consequences for the evolution of the function concept. Its major effect was to extend that concept to include:

(a) Functions defined piecewise by analytic expressions in different intervals. (Thus,

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

was now, for the first time, considered to be a bona fide function.)

(b) Functions drawn freehand and possibly not given by any combination of analytic expressions.

As Lützen [17] put it:

> D'Alembert let the concept of function limit the possible initial values, while Euler let the variety of initial values extend the concept of function. We thus see that this extension of the concept of function was *forced* upon Euler by the physical problem in question.

To see how Euler's own view of functions evolved over a period of several years, compare the definition of function he gave in his 1748 *Introductio* with the following definition given in 1755, in which the term "analytic expression" does not appear [23, pp. 72–73]:

> If, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities. This is a very comprehensive notion and comprises in itself all the modes through which one quantity can be determined by others. If, therefore, x denotes a variable quantity then all the quantities which depend on x in any manner whatever or are determined by it are called its functions ...

Euler's view of functions was reinforced later in that century by work in partial differential equations:

> The work of Monge in the 1770s, giving a geometric interpretation to the integration of partial differential equations, seemed to provide a conclusive proof of the fact that functions 'more general than those expressed by an equation' were legitimate mathematical objects ... [22, p. 86].

See [3], [4], [9], [16], [18], [19], [22], [27] for details on section 3.

4. Fourier and Fourier Series. Fourier's work on heat conduction (submitted to the Paris Academy of Sciences in 1807, but published only in 1822 in his classic *Analytic Theory of Heat*) was a revolutionary step in the evolution of the function concept. Fourier's main result of 1822 was the following.

Theorem. *Any function $f(x)$ defined over $(-\ell, \ell)$ is representable over this interval by a series of sines and cosines,*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right],$$

where the coefficients a_n and b_n are given by

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\frac{n\pi t}{\ell}\,dt \quad \text{and} \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin\frac{n\pi t}{\ell}\,dt.$$

Fourier's announcement of this result met with incredulity. It upset several tenets of 18th-century mathematics. The result was known to Euler and Lagrange (among others), but only for certain functions. Fourier, of course, claimed that it is true for *all* functions, where the term "function" was given the most general contemporary interpretation:

> In general, the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x, there are an equal number of ordinates $f(x)$. All have actual numerical values, either positive or negative or null. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity [23, p. 73].

Fourier's "proof" of his theorem was loose even by the standards of the early 19th century. In fact, it was formalism in the spirit of the 18th century—"a play upon symbols in accordance with accepted rules but without much or any regard for content or significance" [16, p. 33]. To convince the reluctant mathematical community of the reasonableness of his claim, Fourier needed to show that:

(a) The coefficients of the Fourier series can be calculated for *any* $f(x)$

(b) *Any* function $f(x)$ can be represented by its Fourier series in $(-\ell, \ell)$.[6]

He showed this by:

(a′) Interpreting the coefficients a_n and b_n in the Fourier series expansion of $f(x)$ as areas (which made sense for "arbitrary" functions $f(x)$, not necessarily given by analytic expressions)

(b′) Calculating the a_n and b_n (for small values of n) for a great variety of functions $f(x)$, and noting the close agreement in $(-\ell,\ell)$ (but not outside that interval) between the initial segments of the resulting Fourier series and the functional values of $f(x)$.

Fourier accomplished all this using mathematical reasoning that would be clearly unacceptable to us today. However,

> It was, no doubt, partially because of his very disregard for rigor that he was able to take conceptual steps which were inherently impossible to men of more critical genius [16, p. 33].

Fourier's work raised the analytic (algebraic) expression of a function to at least an equal footing with its geometric representation (as a curve). His work had a fundamental and far-reaching impact on subsequent developments in mathematics. (For example, it forced mathematicians to reexamine the notion of integral, and was the starting point of the researches that led Cantor to his creation of the theory of sets.) As for its impact on the evolution of the function concept, Fourier's work:

- Did away with the "article of faith" held by 18th-century mathematicians. (Thus, it was now clear that two functions given by different analytic expressions can agree on an interval without necessarily agreeing outside the interval.)

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[6] Fourier was among the first to highlight the issue of convergence of series, which was of little concern to mathematicians of the 18th century.

- Showed that Euler's concept of "discontinuous" was flawed. (Some of Euler's discontinuous functions were shown to be representable by a Fourier series—an analytic expression—and were thus continuous in Euler's sense.)
- Gave renewed emphasis to analytic expressions.

As we shall see, all this forced a re-evaluation of the function concept. See [3], [6], [7], [9], [16], [19] for details.

As we have noted, the period 1720–1820 was characterized by a development and exploitation of the tools of the calculus bequeathed by the 17th century. These tools were employed in the solution of important "practical" problems (e.g., the Vibrating-String Problem, the Heat-Conduction Problem). These problems, in turn, clamored for attention to important "theoretical" concepts (e.g., function, continuity, convergence). A new subject—analysis—began to take form, in which the concept of function was central. But both the subject and the concept were still in their formative stages. It was a period of "formalism" in analysis—formal manipulations dictated the "rules of the game," with little concern for rigor. The concept of function was in a state of flux—an analytic expression (an "arbitrary" formula), then a curve (drawn freehand), and then again an analytic expression (but this time a "specific" formula, namely a Fourier series). Both the subject of analysis (certainly its basic notions) and the concept of function were ripe for a reevaluation and a reformulation. This is the next stage in our development.

5. Dirichlet's Concept of Function Dirichlet was one of the early exponents of the critical spirit in mathematics ushered in by the 19th century (others were Gauss, Abel, Cauchy). He undertook a careful analysis of Fourier's work to make it mathematically respectable. The task was not simple:

> To make sense out of what he [Fourier] did took a century of effort by men of "more critical genius," and the end is not yet in sight [4, p. 263].

Fourier's result that any function can be represented by its Fourier series was, of course, incorrect. In a fundamental paper of 1829, Dirichlet gave sufficient conditions for such representability:

Theorem. *If a function f has only finitely many discontinuities and finitely many maxima and minima in* $(-\ell, \ell)$, *then f may be represented by its Fourier series on* $(-\ell, \ell)$. *(The Fourier series converges pointwise to f where f is continuous, and to* $[f(x+) + f(x-)]/2$ *at each point x where f is discontinuous.)*

For a mathematically rigorous proof of this theorem, one needed (a) clear notions of continuity, convergence, and the definite integral, and (b) clear understanding of the function concept. Cauchy contributed to the former, and Dirichlet to the latter. We first turn very briefly to Cauchy's contributions.

Cauchy was one of the first mathematicians to usher in a new spirit of rigor in analysis. In his famed *Cours d'Analyse* of 1821 and subsequent works, he rigorously defined the concepts of continuity, differentiability, and integrability of a function in terms of limits.[7]

[7] It should be noted that standards of rigor have changed in mathematics (not always from less rigor to more), and that Cauchy's rigor is not ours. Kitcher [14] suggests that Cauchy's motivation in rigorizing the basic concepts of the calculus came from work in Fourier series. See also [8] for background to Cauchy's work in analysis.

(Bolzano had done much of this earlier, but his work went unnoticed for fifty years.) In dealing with continuity, Cauchy addresses himself to Euler's conceptions (footnote 4) of "continuous" and "discontinuous." He shows that the function

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

(which Euler considered discontinuous) can also be written as $f(x) = \sqrt{x^2}$, and

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{x^2}{x^2 + t^2}\, dt,$$

which means that $f(x)$ is also continuous in Euler's sense. This paradoxical situation, Cauchy claims, cannot happen when *his* definition of continuity is used.

Cauchy's conception of function is not very different from that of his predecessors:

> When the variable quantities are linked together in such a way that, when the value of one of them is given, we can infer the values of all the others, we ordinarily conceive that these various quantities are expressed by means of one of them which then takes the name of *independent variable*; and the remaining quantities, expressed by means of the independent variable, are those which one calls the *functions* of this variable [3, p. 104].

Although Cauchy gives a rather general definition of a function, his subsequent comments suggest that he had in mind something more limited (see [10, p. 10]). He classifies functions as "simple" and "mixed." The "simple functions" are $a + x$, $a - x$, ax, a/x, x^a, a^x, log x, sin x, cos x, arcsin x, arc cos x; and the "mixed functions" are composites of the "simple" ones—say, log(sin x). See [3], [6], [8], [9], [12], [14] for Cauchy's contribution.

Now let us consider Dirichlet's definition of function:

> y is a function of a variable x, defined on the interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established [19].

The novelty in Dirichlet's conception of function as an arbitrary correspondence lies not so much in the definition as in its application. Mathematicians from Euler through Fourier to Cauchy had paid lip service to the "arbitrary" nature of functions; but in practice they thought of functions as analytic expressions or curves. Dirichlet was the first to take seriously the notion of function as an arbitrary correspondence (but see [3, p. 201]). This is made abundantly clear in his 1829 paper on Fourier series, at the end of which he gives an example of a function (the *Dirichlet* function),

$$D(x) = \begin{cases} c, & x \text{ is rational} \\ d, & x \text{ is irrational,} \end{cases}$$

that does not satisfy the hypothesis of his theorem on the representability of a function by a Fourier series (see[10, p. 15]). The Dirichlet function:

- was the first explicit example of a function that was not given by an analytic expression (or by several such), nor was it a curve drawn freehand;

- was the first example of a function that is discontinuous (in our, not Euler's sense) *everywhere*;
- illustrated the concept of function as an arbitrary pairing.

Another important point is that Dirichlet, in his definition of function, was among the first to restrict explicitly the domain of the function to an interval; in the past, the independent variable was allowed to range over all real numbers. See [3], [5], [9], [10], [15], [17], [27] for details about Dirichlet's work.

6. "Pathological" Functions. With his new example $D(x)$, Dirichlet "let the genie escape from the bottle." A flood of "pathological" functions, and classes of functions, followed in the succeeding half century. Certain functions were introduced to test the domain of applicability of various results (e.g., the "Dirichlet function" was introduced in connection with the representability of a function in a Fourier series). Certain classes of functions were introduced in order to extend various concepts or results (e.g., functions of bounded variation were introduced to test the domain of applicability of the Riemann integral).

The character of analysis began to change. Since the 17th century, the processes of analysis were assumed to be applicable to "all" functions, but it now turned out that they are restricted to particular *classes* of functions. In fact, the investigation of various classes of functions—such as continuous functions, semi-continuous functions, differentiable functions, functions with nonintegrable derivatives, integrable functions, monotonic functions, continuous functions that are not piecewise monotonic—became a principal concern of analysis. (One example is Dini's study of continuous nondifferentiable functions, for which he defined the so-called Dini derivatives.) Whereas mathematicians had formerly looked for order and regularity in analysis, they now took delight in discovering exceptions and irregularities. The towering personalities connected with these developments were Riemann and Weierstrass, although many others made important contributions (e.g., du Bois Reymond and Darboux).

The first major step in these developments was taken by Riemann in his *Habilitationsschrift* of 1854, which dealt with the representation of functions in Fourier series. As we recall, the coefficients of a Fourier series are given by integrals. Cauchy had developed his integral only for continuous functions, but his ideas could be extended to functions with finitely many discontinuities. Riemann extended Cauchy's concept of integral and thus enlarged the class of functions representable by Fourier series. This extension (known today as the Riemann integral) applies to functions of bounded variation, a much broader class of functions than Cauchy's continuous functions. Thus, a function can have infinitely many discontinuities (which can be dense in any interval) and still be Riemann-integrable.[8] Riemann gave the following example (published in 1867) in his *Habilitationsschrift*:

$$f(x) = 1 + \frac{(x)}{1^2} + \frac{(2x)}{2^2} + \frac{(3x)}{3^2} + \cdots,$$

where for any real number α the function (α) is defined as 0 if $\alpha = 1/2 + k$

[8] There are, of course, restrictions on the discontinuities of a Riemann-integrable function. As we now know (following Lebesgue), a function is Riemann-integrable if and only if its discontinuities form a set of Lebesgue measure zero.

(k, an integer), and α minus the nearest integer when $\alpha \neq 1/2 + k$ (k, integral). This function is discontinuous for all $x = m/2n$, where m is an integer relatively prime to $2n$ (see [6, p. 325]). In contrast to Dirichlet's function $D(x)$, this one is given by an analytic expression and is Riemann-integrable.

Riemann's work may be said to mark the beginning of a theory of the mathematically discontinuous (although there are isolated examples in Fourier's and Dirichlet's works). It planted the discontinuous firmly upon the mathematical scene. The importance of this development can be inferred from the following statement of Hawkins [10, p. 3]:

> The history of integration theory after Cauchy is essentially a history of attempts to extend the integral concept to as many discontinuous functions as possible; such attempts could become meaningful only after existence of highly discontinuous functions was recognized and taken seriously.

In 1872, Weierstrass startled the mathematical community with his famous example of a continuous nowhere-differentiable function

$$f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x),$$

where a is an odd integer, b a real number in $(0, 1)$, and $ab > 1 + 3\pi/2$ (see [12, p. 387]). (Bolzano had given such an example in 1834, but it went unnoticed.) This example was contrary to all geometric intuition. In fact, up to about 1870, most books on the calculus "proved" that a continuous function is differentiable except possibly at a finite number of points! (See [10, p. 43].) Even Cauchy believed that.[9]

Weierstrass' example began the disengagement of the continuous from the differentiable in analysis. Weierstrass' work (and others' in this period) necessitated a reexamination of the foundations of analysis and led to the so-called arithmetization of analysis, in which process Weierstrass was a prime mover. As Birkhoff notes [3, p. 71]:

> Weierstrass demonstrated the need for higher standards of rigor by constructing *counterexamples* to plausible and widely held notions.

Counterexamples play an important role in mathematics. They illuminate relationships, clarify concepts, and often lead to the creation of new mathematics. (An interesting case study of the role of counterexamples in mathematics can be found in the book *Proofs and Refutations* by I. Lakatos.) The impact of the developments we have been describing was, as we already noted, to change the character of analysis. A new subject was born—the theory of functions of a real variable. Hawkins [10, p. 119] gives a vivid description of the state of affairs:

> The nascent theory of functions of a real variable grew out of the development of a more critical attitude, supported by numerous counterexamples, towards the

[9] The malaise in the understanding and use of the function concept around this time can be gathered from the following account by Hankel (in 1870) concerning the function concept as it appears in the "better textbooks of analysis" (Hankel's phrase): "One [text] defines function in the Eulerian manner; the other that y should change with x according to a rule, without explaining this mysterious concept; the third defines them as Dirichlet; the fourth does not define them at all; but everyone draws from them conclusions that are not contained therein" [17]. See also [3, p. 198].

> reasoning of earlier mathematicians. Thus, for example, continuous nondifferentiable functions, discontinuous series of continuous functions, and continuous functions that are not piecewise monotonic were discovered. The existence of exceptions came to be accepted and more or less expected. And the examples of nonintegrable derivatives, rectifiable curves for which the classical integral formula is inapplicable, nonintegrable functions that are the limit of integrable functions, Harnack-integrable derivatives for which the Fundamental Theorem II is false, and counterexamples to the classical form of Fubini's Theorem appear to have been received in this frame of mind. The idea, as Schoenflies put it in his report ... , was to proceed, as in human pathology, to discover as many exceptional phenomena as possible in order to determine the laws according to which they could be classified.

It should be pointed out, however, that not everyone was pleased with these developments (at least in analysis), as the following quotations from Hermite (in 1893) and Poincaré (in 1899), respectively, attest [15, p. 973]:

> "I turn away with fright and horror from this lamentable evil of functions which do not have derivatives."

> "Logic sometimes makes monsters. For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. Indeed, from the point of view of logic, these strange functions are the most general; on the other hand those which one meets without searching for them, and which follow simple laws appear as a particular case which does not amount to more than a small corner.
>
> In former times when one invented a new function it was for a practical purpose; today one invents them purposely to show up defects in the reasoning of our fathers and one will deduce from them only that.
>
> If logic were the sole guide of the teacher, it would be necessary to begin with the most general functions, that is to say with the most bizarre. It is the beginner that would have to be set grappling with this teratologic museum."

The effect of the events we have been describing on the function concept can be summarized as follows. Stimulated by Dirichlet's conception of function and his example $D(x)$, the notion of function as an arbitrary correspondence is given free rein and gains general acceptance; the geometric view of function is given little consideration. (Riemann's and Weierstrass' functions could certainly not be "drawn," nor could most of the other examples given during this period.) After Dirichlet's work, the term "function" acquired a clear meaning independent of the term "analytic expression." During the next half century, mathematicians introduced a large number of examples of functions in the spirit of Dirichlet's broad definition, and the time was ripe for an effort to determine which functions were actually describable by means of "analytic expressions", a vague term in use during the previous two centuries. See [3], [10], [14], [15] for details of this period.

7. Baire's Classification Scheme. The question whether every function in Dirichlet's sense is representable analytically was first posed by Dini in 1878 (see [5, p. 31]). Baire had undertaken to give an answer in his doctoral thesis of 1898. The very notion of analytic representability had to be clarified, since it was used in the past in an informal way. Dini himself used it vaguely, asking "if every function can be expressed analytically, for all values of the variable in the interval, by a finite or infinite series of operations ("opérations du calcul") on the variable" [5, p. 32].

The starting point for Baire's scheme was the Weierstrass Approximation Theorem (published in 1885): *Every continuous function* $f(x)$ *on an interval* $[a, b]$ *is a uniform limit of polynomials on* $[a, b]$. Baire called the class of continuous functions *class* 0. Then he defined the functions of *class* 1 to be those that are not in class 0, but which are (pointwise) limits of functions of class 0. In general, the functions of *class* m are those functions which are not in any of the preceding classes, but are representable as limits of sequences of functions of class $m - 1$. This process is continued, by transfinite induction, to all ordinals less than the first uncountable ordinal Ω. (Since the Baire functions thus constructed are closed under limits, nothing new results if this process is repeated.) This classification into Baire classes $\alpha/(\alpha < \Omega)$ is called the *Baire classification*, and the functions which constitute the union of the Baire classes are called *Baire functions*.

Baire called a function *analytically representable* if it belonged to one of the Baire classes. Thus, a function is analytically representable (in Baire's sense) if it can be built up from a variable and constants by a finite or denumerable set of additions, multiplications, and passages to pointwise limits.

The collection of analytically representable functions (Baire functions) is very encompassing. For example, discontinuous functions representable by Fourier series belong to class 1. Thus, functions representable by Fourier series constitute only a part of the totality of analytically representable functions. (Recall Fourier's claim that *every* function can be represented by a Fourier series!) As another example, Baire showed that the "pathological" Dirichlet function $D(x)$ is of class 2, since

$$D(x) = \begin{Bmatrix} c, & x \text{ is rational} \\ d, & x \text{ is irrational} \end{Bmatrix} = (c - d) \lim_{n\to\infty} \lim_{m\to\infty} (\cos n!\pi x)^{2m} + d.$$

Moreover, any function obtained from a variable and constants by an application of the four algebraic operations and the operations of analysis (such as differentiation, integration, expansion in series, use of transcendental functions)—the kind of function known in the past as an "analytic expression"—was shown to be analytically representable.

Lebesgue pursued these studies and showed (in 1905) that each of the Baire classes is nonempty, and that the Baire classes do not exhaust all functions.[10] Thus, Lebesgue established that there are functions which are not analytically representable (in Baire's sense). This he did by actually exhibiting a function outside the Baire classification,

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[10] In fact, there are (Lebesgue-) measurable functions which are not Baire functions. At the same time, Lebesgue showed that to every measurable function f there corresponds a Baire function which differs from f only on a set of measure zero.

"using a profound but extremely complex method" [19].[11] According to Luzin [19], "the impact of Lebesgue's discovery was just as stunning as that of Fourier in his time." See [5], [19], [20], [21] for details.

Not all functions in the sense of Dirichlet's conception of function as an arbitrary correspondence are analytically representable (in the sense of Baire), although it is (apparently) very difficult to produce a specific function that is not. Do such nonanalytically representable functions "really" exist? This is part of our story in the next section.

8. Debates about the Nature of Mathematical Objects. Function theory was characterized by some at the turn of the 20th century as "the branch of mathematics which deals with counterexamples." This view was not universally applauded, as the earlier quotations from Hermite and Poincaré indicate. In particular, Dirichlet's general conception of function began to be questioned. Objections were raised against the phrase in his definition that "it is irrelevant in what way this correspondence is established." Subsequently, the arguments for and against this point linked up with the arguments for and against the axiom of choice (explicitly formulated by Zermelo in 1904) and broadened into a debate over whether mathematicians are free to create their objects at will.

There was a famous exchange of letters in 1905 among Baire, Borel, Hadamard, and Lebesgue concerning the current logical state of mathematics (see [5], [20], [21] for details). Much of the debate was about function theory—the critical question being whether a definition of a mathematical object (say a number or a function), however given, legitimizes the existence of that object; in particular, whether Zermelo's axiom of choice is a legitimate mathematical tool for the definition or construction of functions. In this context, Dirichlet's conception of function was found to be too broad by some (e.g., Lebesgue) and devoid of meaning by others (e.g., Baire and Borel), but was acceptable to yet others (e.g., Hadamard). Baire, Borel, and Lebesgue supported the requirement of a definite "law" of correspondence in the definition of a function. The "law," moreover, must be reasonably explicit—that is, understood by and communicable to anyone who wants to study the function.

To illustrate the point, Borel compares the number π (whose successive digits can be unambiguously determined, and which he therefore regards as well defined) with the number obtained by carrying out the following "thought experiment." Suppose we lined up infinitely many people and asked each of them to name a digit at random. Borel claims that, unlike π, this number is not well defined since its digits are not related by

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[11] The construction is quite "messy" and uses the axiom of choice. Using nonconstructive arguments, one can show by a counting argument that the Baire functions have cardinality c. Since the set of all functions has cardinality 2^c, there are uncountably many functions which are not analytically representable in Baire's sense.

Baire's notion of analytic representability is not the last word on the subject. Luzin [19] mentions the example of an "analytic expression"

$$f(x) = \overline{\lim_{y\to\infty}}\ \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} P_{m,n}(x, y)$$

which, for a suitable choice of the polynomials $P_{m,n}(x, y)$, is not expressible as a Baire function.

any law. This being so, two mathematicians discussing this number will never be certain that they are talking about the *same* number. Put briefly, Borel's position is that without a definite law of formation of the digits of an infinite decimal, one cannot be certain of its identity.

Hadamard had no difficulty in accepting as legitimate the number resulting from Borel's thought experiment. By way of illustration, he alluded to the kinetic theory of gases, where one speaks of the velocities of molecules in a given volume of gas although no one knows them precisely. Hadamard felt that "the requirement of a law that determines a function . . . strongly resembles the requirement of an *analytic expression* for that function, and that this is a throwback to the 18th century" [19].

The issues described here were part of broad debates about various ways of doing analysis—synthetic versus analytic, or idealist versus empiricist. These debates, in turn, foreshadowed subsequent "battles" between proponents and opponents of the various philosophies of mathematics (e.g., formalism and intuitionism) dealing with the nature and meaning of mathematics. And, of course, the issue has not been resolved.[12] See [4], [5], [19], [20], [21] for details.

The period 1830–1910 witnessed an immense growth in mathematics, both in scope and in depth. New mathematical fields were formed (complex analysis, algebraic number theory, non-Euclidean geometry, abstract algebra, mathematical logic), and older ones were deepened (real analysis, probability, analytic number theory, calculus of variations). Mathematicians felt free to create their systems (almost) at will, without finding it necessary to seek motivation from or applications to concrete (physical) settings. At the same time there was, throughout the 19th century, a reassessment of gains achieved, accompanied by a concern for the foundations of (various branches of) mathematics. These trends are reflected in the evolution of the notion of function. The concept unfolds from its modest beginnings as a formula or a geometric curve (18th and early 19th centuries) to an arbitrary correspondence (Dirichlet). This latter idea is exploited throughout the 19th century by way of the construction of various "pathological" functions. Toward the end of the century, there is a reevaluation of past accomplishments (Baire classification, controversy relating to use of the axiom of choice), much of it in the broader context of debates about the nature and meaning of mathematics.

9. Recent Developments. Here we briefly touch on three more recent developments relating to the function concept.

A) L_2 *Functions*. The set $L_2 = \{f(x) : f^2(x)$ is Lebesgue-integrable$\}$ forms a "Hilbert space"— a fundamental object in functional analysis. Two functions in L_2 are considered to be the same if they agree everywhere except possibly on a set of Lebesgue measure zero. Thus, in L_2 Function Theory, one can always work with representatives in an equivalence class rather than with individual functions. These notions, as Davis and Hersh observed [4, p. 269],

[12] There has recently been a renewed interest, among others by computer scientists, in Brouwer's "intuitionistic mathematics." The revival, in the form of "constructive mathematics," was led by E. Bishop, and is highlighted in an article by M. Mandelkern, "Constructive Mathematics," Math. Mag. 58 (1985) 272–280.

involve a further evolution of the concept of function. For an element in L_2 is not a function, either in Euler's sense of an analytic expression or in Dirichlet's sense of a rule or mapping associating one set of numbers with another. It is function-like in the sense that it can be subjected to certain operations normally applied to functions (adding, multiplying, integrating). But since it is regarded as unchanged if its values are altered on an arbitrary set of measure zero, it is certainly not just a rule assigning values at each point in its domain.

B) *Generalized Functions* (Distributions). The concept of a distribution or generalized function is a very significant and fundamental extension of the concept of function. The theory of distributions arose in the 1930s and 1940s. It was created to give mathematical meaning to the differentiation of nondifferentiable functions—a process which the physicists had employed (unrigorously) for some time. Thus, Heaviside (in 1893) "differentiated" the function

$$f(x) = \begin{cases} 1, & x > 0 \\ 1/2, & x = 0 \\ 0, & x < 0 \end{cases}$$

to obtain the impulse "function"

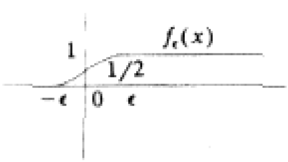
$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}.$$ [13]

(In 1930, Dirac introduced $\delta(x)$ as a convenient notation in the mathematical formulation of quantum theory.)

In formal terms, a *distribution* is a continuous linear functional on a space D of infinitely differentiable functions (called "test functions") that vanish outside some interval $[a, b]$. To any continuous (or locally integrable) function F, there corresponds a distribution $\Phi_F: D \to \mathbb{C}$ given by $\Phi_F(x) = \int_{-\infty}^{\infty} F(t)x(t)\,dt$. However, not every distribution comes from such a function: The distribution $\delta: D \to \mathbb{C}$ given by $\delta(x) = x(0)$ corresponds to the "Dirac δ-function" mentioned above, and does not arise from any function F in the way described above. See[4], [18], [26].

A basic property of distributions is that each distribution has a derivative that is again a distribution.[14] In fact,

[13] The following is a heuristic argument: Approximate $f(x)$ by a sequence of differentiable functions $f_\epsilon(x)$ as in the diagram:



; then $f'_\epsilon(x) \to \delta(x)$ as $\epsilon \to 0$.

[14] In particular, every continuous function is "differentiable" (that is, has a distribution as its "derivative"). In fact, L. Schwartz, one of the creators of the theory of distributions, claimed that he had introduced distributions to be able to differentiate continuous functions. Lützen [18, p. 305] asserts that "the theory of distributions probably constitutes the closest approximation to Euler's vision of a generalized calculus," a vision that Euler tried to put into practice in his solution of the Vibrating-String Problem.

> The enduring merit of distribution theory has been that the basic operations of analysis, differentiation and convolution, and the Fourier/Laplace transforms and their inversion, which demanded so much care in the classical framework, could now be carried out without qualms by obeying purely algebraic rules [26, p. 338].

C) *Category Theory*. The notion of a function as a mapping between arbitrary sets gradually became dominant in the mathematics of the 20th century.[15] Algebra had a major impact on this development, in which the concept of a function was placed in the general framework of the concept of a mapping from one set into another. Thus, linear transformations of vector spaces (principally, $\mathbb{R}^n$ and $\mathbb{C}^n$) were dealt with throughout much of the 19th century. Homomorphisms of groups and automorphisms of fields were introduced in the latter part of that century. As early as 1887, Dedekind gave a fairly "modern" definition of the term "mapping" [23, p. 75]:

> By a mapping of a system S a law is understood, in accordance with which to each determinate element s of S there is associated a determinate object, which is called the image of s and is denoted by $\varphi(s)$; we say too, that $\varphi(s)$ corresponds to the element s, that $\varphi(s)$ is caused or generated by the mapping φ out of s, that s is transformed by the mapping φ into $\varphi(s)$.

Analysis, too, played a major role in this extension of the domain and range of definition of a function to arbitrary sets. (Recall that Dirichlet's definition of function was as an arbitrary correspondence between (real) *numbers*.) Thus, Euler and others in the 18th century treated (informally) functions of several variables. In 1887, considered the year of birth of functional analysis, Volterra defined the notion of a "functional" which he called a " function of functions." (A *functional* is a function whose domain is a set of functions and whose range is the real or complex numbers.) In the first two decades of the 20th century, the notions of metric space, topological space, Hilbert space, and Banach space were introduced; functions (operators, linear operators) between such spaces play a prominent role. See [15] for details.

In 1939, Bourbaki gave the following definition of a function [3, p. 7]:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a *functional relation* in y if, for all $x \in E$, there exists a unique $y \in F$ which is in the given relation with x.
>
> We give the name of *function* to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with x; y is said to be the *value* of the function at the element x, and the function is said to be *determined* by the given functional relation. Two equivalent functional relations determine the *same* function.

Bourbaki then also gave the definition of a function as a certain subset of the Cartesian product $E \times F$. This is, of course, the definition of function as a set of ordered pairs.

All of these "modern" general definitions of function were given in terms of sets, and hence their logic must receive the same scrutiny as that of set theory.

In category theory, which arose in the late 1940s to give formal expression to certain aspects of homology theory, the concept of function assumes a fundamental role. It can be described as an "association" from an "object" A to another "object" B. The

[15] "Naive" set theory was developed by Cantor during the last three decades of the 19th century.

"objects" A and B need not have any elements (that is, they need not be sets in the usual sense). In fact, the objects A and B can be entirely dispensed with. A "category" can then be defined as consisting of functions (or "maps"), *which are taken as undefined (primitive) concepts* satisfying certain relations or axioms. In fact, in 1966 Lawvere outlined how category theory can replace set theory as a foundation for mathematics. See [11] for details.

In the recent developments outlined in this section, we have seen the function concept modified (L_2 functions), generalized (distributions), and finally "generalized out of existence" (category theory). Have we come full circle?

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