The Development of Mathematics, in a Nutshell

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Abstract

The development of mathematics is intimately interwoven with the progress of civilization, influencing the course of history through its application to science and technology.

But mathematics has changed. Even the mathematics of the 1800s can seem quite strange now, so greatly has mathematics evolved in the past 100 years and so thoroughly has it been reworked in the post-modern approach.

Despite its arcane appearance from the outside looking in, the present, abstract and highly specialized state of mathematics is the natural evolution of the subject, and there is much ahead that is exciting.

Though mathematical knowledge is ancient, stretching back to the Stone Age, the evolution of mathematics to its current modern state has seen fundamental changes in concepts, organization, scope, outlook, and practice. Without understanding the evolution of mathematical thought, it is difficult to appreciate modern mathematics in its contemporary, highly specialized state.

Here, then, is the story of mathematics, in a nutshell.

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Seven Periods of Mathematical Practice

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Roughly speaking, I would identify seven periods in the evolution of mathematics, each with distinct characteristics.

1. Proto-Mathematics (from the mists of ancient time, through the archeological evidence of c.30000 BCE, up to 2000 BCE): empirical, not abstract, basic

2. Ancient Mathematics (from 2000 BCE up to 800 BCE): empirical, number and figures abstracted, highly sophisticated (Babylonian, Egyptian), not axiomatic

3. Classical Mathematics (from 800 BCE to 1500 CE): axiomatic geometry (Greek), highly sophisticated geometry, sophisticated abstraction in algebra and algorithmization of arithmetic (Indian, Arabic, Central Asian)

4. Mercantile Mathematics (from 1400 CE to 1500 CE): improvement in numeration, symbolic development, and symbolic shorthand arithmetic (Renaissance Europe), sophisticated algebra and solution of equations (Italian wranglers)

5. Pre-Modern Mathematics (from 1500 CE to 1700 CE): functions, continuous mathematics, analytic geometry, calculus, applications to science

6. Modern Mathematics (from 1700 CE to 1950 CE): modern abstract analysis, modern abstract algebra, modern abstract geometry, modern logic – all freed mathematics from the perspectives, paradoxes, and problems encountered during the classical and mercantile periods

7. Post-Modern Mathematics (from 1950 CE to present): dramatic expansion in scope and productivity in mathematics, based upon axiomatic methods, accelerated by unprecedented growth in science, applied science, engineering, technology, statistics, and applications to all areas of human endeavor.

Proto-Mathematics

The essence of mathematics, call it proto-mathematics, exists in empirical observations and interactions with the environment.

Even the earliest man had need of basic mathematical understanding: counting, keeping time, shape and symmetry in crafts and art, and the practical matters of numerical communication, measuring and building, albeit roughly.

Over the course of many millennia, mankind evolved into a more settled lifestyle involving the cultivation of land and livestock. This meant greater food with less work per capita, the impetus for greater specialization (crafts), the growth of communities, the development of classes and hierarchies (warrior, farmer), the growth of administration, and greater leisure. Writing allowed man to transmit his knowledge, to teach, and learn, and preserve what he had learned from generation to generation.

Ancient Mathematics

From empirical mathematics arose, through abstraction, the sciences of arithmetic (number) and geometry (figure). These were developed into an extremely sophisticated science by the Babylonians and the Egyptians, and reached spectacular heights during their respective civilizations, applied to astronomy, the regulation of time, administration, planning and logistics, land surveying, calculation of areas and volumes, construction, and the engineering of incredible monuments.

> By 3500 B.C.E., the "Egyptians had a fully developed number system that would allow counting to continue indefinitely with only the introduction from time to time of a new symbol." And by 3000 B.C.E., the Babylonians had developed a system of writing from pictographs which included a fully developed sexagesimal positional system and positional notation for sexagesimal fractions. ([Bur], p.11)

The ancient period viewed mathematics as the phenomena of number and extent. Each may have been viewed abstractly, and reasoning given, but the formal structure as a whole was absent. The knowledge and facility of Babylonian and Egyptian mathematics was quite sophisticated: surprisingly accurate astronomical computations, the value of π to several decimal places, formulas for areas of a number of two- and three- dimensional objects, knowledge of structural stability and its manifestation in wondrous architecture (the pyramids of Egypt, the granaries of Mesopotamia, the hanging gardens of Babylon, etc.), solving of quadratic and in some cases cubic equations, place-value positional system of numeration and computation, including positional decimals among the Babylonians, administration of lands and taxes, accurate surveying, the logistics of administrative planning, maintenance of records, and supplying of large armies of soldiers and workers.

Early Classical Mathematics

The Greeks introduced to mathematics a fundamental abstraction: the separation of the proceedings of mathematics from the empirical to the logical, and the arrangement of the facts of geometry upon an hierarchy of statements, pinned upon acceptance of first principles, or axioms.

The view of mathematics was of a formal structure as a whole, held together by the laws of thought, with results organized into a linear body of work, each proved in terms of statements already accepted or proved, with the full understanding of the need for first principles, or axioms.

The science of Geometry flourished under the Greeks, including applications to mechanics, machines, astronomy, and engineering, both Greek and Roman. Many challenging problems in curvilinear and solid geometry were obtained through methods of the Calculus: finding areas and perimeters by a process of finer and finer approximation by summation (though not formally a computation of the limit).

The Encountering of Paradox

In the development of arithmetic and the number concept, the Greeks discovered early on the inadequacy of the common notion of number (rational number) to describe lengths. Indeed, a simple length, the diagonal of a square, eluded their common notion of number.

This was the beginning of the discovery of paradoxes *in the theory* of mathematics. The fact that the diagonal and side of a square are logically incommensurable is not a problem of reality; it is a problem with the logical theory that had been developed: here is this length, very tame, very self-evident. And here is this theory: very appealing, very useful, very valuable, matching reality very well up to this point. And this theory blends arithmetic with geometry, number with measure. But the theory now, irrefutably, has a problem. These lengths are incommensurable. There is no (rational) number that can measure that length, no matter how small the scale of measurement is!

This blew a fuse in the ancient Greek world and led to all kinds of intellectual searching to try to find the flaw, the problem.

The key point to keep in mind, is that the problem is with construction of the mathematical theory. It is NOT an issue with the world, or with progress, or with science, or with engineering. In the real world, diagonals can be measured, no problem. In fact, all lengths can be measured up to the precision of the measuring instrument being used. Which means that all measurements are rational, and there is no practical difficulty.

But it was deeply unsatisfying for the Greeks to have a *theory* in which every length cannot be represented by some "number". Given the complexities of the concept of number, trouble in attempting to expand it to cover all measurement (existence of irrationals, etc.), and the paradoxes of number, space, and time of Zeno and others, geometry was viewed as the rock on which mathematical reality rested. Numbers were regarded as useful, but with suspicion and not always reliable.

This way of thinking led to geometry being supreme to the Greeks. And the towering achievement of Euclid's presentation of the Elements of Geometry kept that position for Geometry through to the end of the 1700s and into the early 1800s.

But now a separation had clearly occurred between concrete and abstract mathematics, between practical science and engineering, and theoretical mathematics.

Aside: The Resolution of the Paradox The remedy for the problem of numbers is its expansion to include all (Cauchy) sequences of rational numbers, since these are convergent so long as the point of their convergence existed. In this way can be filled in $\sqrt{2}, \sqrt{3}, \sqrt{p}$, for all primes p, and, in principle, all numbers that can be approximated to indefinite precision (i.e. have a decimal expansion or iterative/inductive formula). These are the "real" numbers, and their establishment and properties is the provenance of analysis fundamentals, an accomplishment that was finally completed in the 1800s by Cantor, Dedekind, and others. This new and *much* larger domain of numbers is no longer a countable infinity but an uncountable infinity of numbers, as shown by Cantor.

Late Classical Mathematics

Algebra, the science of equations, was already well developed in Babylonian and Egyptian times. But it flourished during the Islamic era under Arabic and Central Asian mathematicians, as well as under the Indian mathematicians. Here it was that the modern notions of solution of algebraic expressions was developed into an algorithmic process by the Arabic and Central Asian mathematicians, and applied to astronomy, optics, engineering, and commerce.

Mercantile Mathematics

A flourishing trade and financial system had emerged during the thousand or so years of Islamic rule, first under the Baghdad and Damascus caliphs, then under the over-lordship of the Mongols, and finally under the courts of the Seljuk Turks. Computation, calculation, and other such practical mathematical matters, including negative numbers, were developed and flourishing in Arabia, Central Asia, India, and China.

With the quickening of learning again in Europe during the Renaissance and the rise of the merchant states of Italy after the crusades, the mercantile mathematics of the Middle East and East arrived to Europe to revive arithmetic knowledge and the practical arts of computation.

Under the resurgent interest in mathematics introduced in the mercantile period, further developments arose in arithmetic and algebra: symbolism was introduced into mathematics, and the challenge of finding solutions to polynomials of order 3, 4, and 5 was tackled. Third and fourth degree polynomials were solved by radicals. The challenge was for higher degree polynomials.

The Rise of the Notion of Functions

It is at this point that the next major innovation is made in mathematics, one that unites arithmetic, geometry, algebra, and analysis. That notion is the notion of continuous function, its use in modeling physical and geometric situations, and its manipulations and analysis using algebra and arithmetic. This approach has been enormously fruitful, expanding the range of mathematics to all of science.

The notion of function was developed out of the empirical observations and modeling, using functions, by Galileo, and its applications to the problems of geometry, analytic geometry, by Descartes and Fermat. The notions were deepened through the development of the analytic functions of trigonometry, logarithms, and exponential functions (expanding the stable of functions away from the algebraic polynomials, radicals, and rational functions of classical algebra). These developments led to the watershed results of the calculus, namely the unification of the differential calculus (problem of tangents), and the integral calculus (problem of areas), and their applications in optimization, physics, and all manner of areas now rendered possible.

The Pre-Modern Period

Pre-modern mathematics is the relaxing of the synthetic classical geometry with the enhancement of the analytic geometrical methods and the rise of a symbolical algebra. The needs generated by the analytic methods, together with improvements in symbolism, led to greater attention to and progress in what I would call "classical" algebra, which at this time was really the theory of equations, polynomials. Also, there was classical number theory without modern algebra, classical geometric analysis without limits or the infinitesimal calculus, classical complex numbers, classical probability theory.

Negative numbers, now much more familiar due to trade and the progress of arithmetic algorithms, were still viewed with some suspicion and used reluctantly as computing devices that helped to get correct answers even if one temporarily had to suspend the "meaning" of a certain step and simply follow along formally. This view of numbers was bolstered by the presence in computations and solutions of numbers that had no real meaning in the modeled "reality", e.g. negative numbers, roots, imaginary numbers. It is in this context that Euler's advances and the bold use of formal manipulation can be considered quite phenomenal and, in many ways, ahead of his time.

Though the Calculus was there, it was still viewed as a geometrical subject, with the attendant support of numerical computation and methods for derivation of otherwise geometrical phenomena.

Mathematicians, right through the time of Euler in the early 1800s, called themselves "geometers" (Newton, Leibniz, Fermat, L'Hopital, Euler even –

all were geometers, who also studied numbers, science, and other matters). Only in the late 1800s (Gauss, Riemann, the understanding and acceptance of non-Euclidean geometries), did they call themselves "mathematicians" or "logicians").[San06]

One can say that pre-modern mathematics was mathematics roughly up to the end of the 1600s (Fermat, Bernoulli, Leibniz, Newton), and perhaps middle to late 1700s. Euler was a transitional figure over the dividing line with modern mathematics during the first part of the 1700s (Euler).

Modern Period

The modern period of mathematics was characterized by the comprehensive and systematic synthesis of mathematical knowledge. It is remarkable for its uncovering of deep structural phenomena, and the generalization, unification, and synthesis of all of mathematics.

Modern mathematics can be said to have been born in the 1800s, and characterized by grappling with the challenges from the Classical period, as well with addditional disturbances that had been found and continued to be found with the theory of mathematics as then understood: the basis of the integral and differential calculus, the impossibility of a solution by radicals of polynomials of degree five or higher (which explains why the classical geometric problems had no solution), paradoxes in logical foundations (Russell, Burale Forte, etc.), shocking results about higher orders of infinity and Cantor's theory of sets (the Continuum Hypothesis), the "monsters" of real analysis functions and measure theory (continuous but nowhere differentiable functions, etc.), and the shocking limitations of logic in Godel's Incompleteness Theorems.

What resulted was a rich development and re-working of mathematics:

- the Galois theory, that resolved as impossible the unsolved problems from classical geometry and also the unsolved problems from classical algebra and theory of equations;

- the careful definition of the concept of limit, the treatment of infinite series as a limit of partial sums, and the foundation of analysis on arithmetical terms, i.e. the construction of the real number system as equivalence classes of Cauchy sequences, thus effectively completing the number system and including the irrational numbers;

- the investigation of algebraic structure of integers, polynomials, number theory, of matrices, quaternions, and vectors, modern algebraic structures, and algebraic mathematics applied to geometry and the continuum;

- the resolution of the parallel postulate unsolved problem by the demonstration of logically valid non-Euclidean geometries;

- the establishment of a set theory able to handle the infinite and higher orders of infinity;

- the demonstration of the existence of transcendental numbers, and indeed their dominance among all numbers and the relative small infinity of the rationals and even the algebraic numbers, as well as the demonstration of the transcendence of π and e.

So modern mathematics is modern algebra, Galois theory of algebraic equations, modern number theory, analysis, set theory, complex variables, and Fourier analysis, etc.: much of the content of advanced undergraduate and graduate level mathematics.

Modern mathematics can be said to have been from the mid 1800s to the early middle 1900s, with mathematicians such as Cauchy, Weierstrass, Riemann, Dedekind, Bolzano, Cantor, and Hilbert, all establishing the language and patterns of thinking characteristic of modern mathematics. Though Laplace, Poisson, Gauss, Fourier, and Lagrange contributed to the establishment of many modern areas of investigation, in the late 1700s and through the 1800s, and uncovered important parts of the structures of modern mathematics, the form of their work and the style of their exposition would now appear archaic, being, as it usually was, in the style of pre-modern mathematics.

Modern mathematics, though more unified, abstract, and diverse than the pre-modern mathematics, is still not the mathematics of today. Though the deeper structures of mathematical fields were being uncovered, they were not yet reflected in a standardized approach to its various areas. This is the legacy that has characterized the post-modern period.

Post-Modern Period

Contemporary mathematics is truly vast. The last time when it is said that one man could understand all of mathematics was perhaps in the 1800s. That time has now long since gone and is not likely to return.

The mathematics being practised today looks surprisingly different from the mathematics of even the early part of the 1900s. What has changed? Early in the post-modern period, the presentation of mathematics was thoroughly reworked to reflect the deeper structures that have been discovered to permeate mathematics. Post-modern mathematics is thus characterized by the analytic and set theoretic language of mathematical practice and also by the modern algebraic. Consider topology, modern geometry (very different from classical geometry), and all manner of modern abstractions, most of which are axiomatized, and the proceedings within which are axiomatic.[1]

While it may seem that mathematicians have cast aside any connection with the "real world" and have declared it as unnecessary for the heart of mathematics, this is most decidedly not the case. Yes, there is an unabashed presentation of mathematics in terms of abstract definitions, axiomatized mathematical structures, and the investigation of the resulting objects, systems and their

[1] See *The Phenomenology of Mathematics* ([Ebr06]) for a discussion of axiomatics, its positives and negatives.

properties. But the state of modern abstract mathematics is a continuum along the natural evolution of the subject and body of knowledge.

The opportunity for fruitful application to technology is enormous, and provided that the greater risk of misunderstanding in education can be addressed, there is much ahead that is exciting.

Further Reading

The following provide additional reading on the material of this article: [Ale56], [AKL63], [Bou], [Boya], [Boyb], [Bur], [Cheb], [Chea], [Cro], [Ebr09b], [Ebr09a], [Fef], [Fef98], [Fef99], [Gal94], [Gul97], [Hoy], [Kle86], [Kli], [Lit49], [Pan], [Sub], [Wal06].

This article is posted, in part, at
http://www.mathscitech.org/articles/development-of-mathematics

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