Teaching Mathematics "In Tunic": Thoughts on Exploratory and Topical Mathematics

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The Nonlinearity of Mathematical Ideas

Mathematics is a richly spun tapestry threaded with interconnections from a multiplicity of endeavors, perspectives, and disciplines, both theoretical and applied. Contrary to its typical presentation, mathematics is *not* a linear subject.

For an instructor, this presents a number of challenges:

- how best to address the non-linear, inter-woven nature of mathematical ideas while still maintaining sufficient pace through the material?

- what is the appropriate trade-off between guided exploration, which has the desirable element of personal discovery but takes longer, and between lecture, which allows the coverage of more material but is perhaps less readily assimilated?

- what is the appropriate balance between teaching facts (or techniques) and teaching new ways of thinking about them?[1]

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[1] "The important thing in science is not so much to obtain new facts as to discover new ways of thinking about them." – William Bragg, (Nobel Laureate in Chemistry)

An Exploratory, Topical Approach to Teaching Mathematics

I believe that students who are in a *general* educational program[2] and who are first encountering a mathematical subject, would benefit more from being presented with an exploratory, topical approach than they would from the usual, linearly ordered alternative that begins at the beginning and travels slowly down a long path filled with technical detail.

What I have in mind is the presentation of a mathematical subject with a carefully selected ordering of topics that does the following:

1. it should situate the material in the context of the intellectual history of the subject,

2. it should identify the goal to which a particular effort is directed, and motivate the desirability of this goal,

3. it should recognize that there is drama, excitement, and tension in the build-up of a story around conflict — and that intelletual struggle, suitably situated in the history of a subject, is certainly an interesting conflict!

4. it should identify those points at which guided exploration will enhance understanding, a feeling of ownership, connectedness with the material, and a start toward the development of important technical skills.

Over the years, there have been numerous discussions about the forms of mathematical teaching and their comparative advantages. Whether a class is taught using the lecture format or in seminar style, whether it is taught using the Moore method or one of various modified Moore methods[Cha95]; whatever form is chosen, I believe the preceeding points, when woven into the presentation, enhance the understanding of non-trivial mathematics by general education students.

The Defensive Armor of Mathematics

The preceeding discussion concerns the personal practice of each instructor in preparing to teach mathematics. But in addition to this, there are larger forces that I believe push against a humanistic, topical, and exploratory approach to mathematics education.

One of the problems I see is that over time mathematicians themselves have carefully separated their professional interests from more general inquiries as a defense against the troublesomeness of philosophical, foundational, and tangential questions.

Out of this, there has arisen a public reflex among mathematicians to claim that their work is not as much concerned with the material and concrete as with

[2] A general education program, also called a liberal arts program, as opposed to a technical program.

the logical inter-relation between concepts. Not so much with the "why" of the world but with the "whether" of hypothetical ideas.

Consider the following quote from a prominent member of Bourbaki:

> "On foundations, we believe in the reality of mathematics, but of course when philosophers attack us with their paradoxes we rush to hide behind formalism and say, "Mathematics is just a combination of meaningless symbols," and then we bring out Chapters 1 and 2 on set theory. Finally we are left in peace to go back to our mathematics and [to] do it as we have always done, with the feeling each mathematician has that he is working with something real. [...] That is Bourbaki's attitude toward foundations." - J.A. Dieudonne ([Die70])

Though perhaps more frank than others might express it, I claim that this passage captures the behavior if not the expressed sentiment of more than a few mathematicians.

Such a response is understandable considering those historical periods in which the foundations and practice of mathematics endured intense and painful scrutiny. During the 1500s, the intense scrutiny of the Church led Galileo to endure years of house arrest and finally the public disavowal of his scientific beliefs, though in private he continued to adhere to them. During the period of the informal Calculus in the 1700s, Bishop Berkeley and others carefully scrutinized the practice of mathematical analysis and scathingly denounced mathematicians as professing an ideal of pure logic while practicing in a manner far removed from this. Indeed, Gauss himself refrained from publishing his work on non-Euclidean geometry, fearing that society was not ready for the shattering philosophical consequences of this discovery. As a final example, the renewed scrutiny of mathematical practice, foundations of mathematical reasoning, and the mathematical enterprise itself by Cantor, Bertrand Russell, Frege, Hilbert, Godel, and others in the early 1900s, led to another round of pandemonium over the paradoxes of set theory, the difficulties of the logicist program, and the philosophy of mathematics.

Thus, history provides repeated examples where this public / private separation has been convenient for those practicing the mathematical sciences to be able to continue their work with minimal intrusion, censoring, and distraction.

Impact on Educational Practice

But it is the general education student who is, unfortunately, the "innocent victim" of this split between the private understanding of the mathematician and scientist and the public presentation. Once accepted as "acceptable", the spirit of this public/private separation permits the teaching of mathematics to be considered as a public activity, and therefore to proceed formally, with the private side of mathematics revealed to a comparatively privileged few. The result, I believe, is a self-reinforcing cycle that perpetuates the presentation of sterilized mathematics to so many.

But what becomes of those who have been taught mathematics in the formalized environment of its public practice? Can we blame anyone other than ourselves for the poor publicity that mathematics unfortunately receives? Coopting the form of recent military jingoisms, I would ask: Are we winning the defense against the troublesomeness of having to engage with essentially non-mathematical questions, only to be losing the war on perception, and with it the hearts and minds of the majority of those who pass through our classrooms?

To the average student, mathematics appears dull, somewhat irrelevant, and worst of all, sterile: exercise, computation, and manipulation of fact without apparent goal or the reasonable promise of return. If we begin with the assumption that even average students are intelligent, I believe we should ask ourselves: why should they be any more interested in verifying *unmotivated* facts or performing *unmotivated* computations than in counting grains of sand on the beach?

The operative word in the preceeding discussion is *motivation*, the captivation of interest. At risk of offending the purists in the audience, I am suggesting that we could fruitfully borrow a page out of the playbook of screenwriters who know that there is little that is more captivating to humans than drama. And mathematics, I claim, when presented in the right way, is full of the drama of intellectual struggle.

> "We are an intelligent species and the use of our intelligence quite properly gives us pleasure. In this respect the brain is like a muscle. When it is used we feel very good. Understanding is joyous." – Carl Sagan, Broca's Brain

A Linear, Formalist Presentation, is Easier than the Alternative

A second problem that I believe hinders holistic mathematics education is that it is significantly easier for an instructor to teach linear, formal mathematics, as opposed to exploratory, topical mathematics. Thus, the linear, formal method of presentation is often chosen, despite the fact that this formalist approach is typically poison to student interest, lulling the typical student into a semi-hypnotic trance of "scribing" the stream of symbols and words that pours onto chalk- or white-board.

There is no doubt that generating genuine engagement is difficult and time-consuming. And, if the attempt at student engagegment and broad thinking is successful, there comes with this success the additional time-demand of having to grapple with the difficult questions of students thinking outside the boundaries of "on-topic" questions. Add to this the fact that the understanding and appreciation of mathematical ideas is an amorphous, less easily measurable goal, and the temptation to declare such questions to be "out of bounds" and adopt a linear, formal approach to the "business of presenting facts", is certainly understandable.

And yet, if we succumb to this temptation, what are we really achieving? Are we measuring the ability to learn what is taught, or are we seeking to deepen the understanding of those whom we teach?[3]

Thus, I believe it is not only a problem of the public formalist position of mathematics that has been armored against critics, but also the very real questions of time and societal valuation of good teaching.

Mathematics "In Tunic": Taking off the Defensive Armor

We come now to the title metaphor. By teaching mathematics "in tunic", I am suggesting an approach that uses the four points given at the start of this article, but which require being willing to set aside the linear, formal, logically meticulous armor of publicly presented mathematics.

I would claim that privately, most mathematicians, like most students, are intruiged by the questions of 'why'. I would claim also that privately, every practitioner of science knows that despite significant and ever increasing progress, man has as yet uncovered but a minute portion of nature's workings. Hence the continued enthusiasm and effort in mathematics and the sciences.

We can see in the autobiographical and introspective writings of mathematicians such as Rota ([Rot97]), Davis and Hersh ([DH81]), Lakatos ([Lak76]), Hadamard ([Had]), Poincare ([Poi]), and others, that their private approach to mathematics was often quite different from the public, formal one. Indeed, Gauss himself, one of the first of the modern mathematicians to insist upon rigor, would privately engage in laborious computation and the development of numerous specific examples (putting together a table of 3 million primes before conjecturing that the number of primes less than x asymptotically approaches $x/\log(x)$ – the celebrated Prime Number Theorem). But again, in his public presentations, Gauss often gave no trace of his private explorations, to the extent that no less a mathematician than Abel deplored: "He [Gauss] is like the fox, who erases his tracks in the sand with his tail."[Kle07], p.143.

Toward a Better General Mathematics Curriculum

So the question becomes: how can we get the private energy and enthusiasm of most professional mathematicians and scientists to be reflected in the public curricula and general mathematics education of students?

In more relaxed, private environments, where master-disciple style relations can thrive and when fears of censure at revealing the inner thoughts of the practising professor are less present, we often see mathematics and science being beautifully presented "in tunic" instead of in armor.

We see *imagination* and leaps of faith, intuition, and excitement, conjectures and refutations. We see the dialectic of Lakatos ([Lak76]). And we see "proof" in its proper place in service to mathematical understanding. We see explorations

[3] "The intelligence is proved not by ease of learning, but by understanding what we learn." – Joseph Whitney

of examples, exploratory computations, and the use of automatic tools in the service of these explorations to allow the conjecture/testing cycle to proceed rapidly, able to keep up with the creativity of the search. In this form, we see mathematics *alive* and energetic, inspired, and inspirational. We see students light up, and rally to the search, become engrossed and excited.[4]([Mau], [Kra91])

To teach mathematics in this manner takes time, thought, and encouragement for students to explore. Students, in their turn, need time, guidance, and most importantly, the posing of stimulating problems out of whose exploration the important points of the theory will either drop out naturally or be well motivated. The succeeding presentation of the theory will then seem natural, provide a climax, be much more readily digestible, and provide that link to humanistic interest that all students find irresitable. Such an approach emphasizes the key ways of thinking that allowed some "distinguished mathematical ancestor" to leap the chasm, span the ravine, ford the river.

By the time the module is over, all students will have traveled the terrain, experienced the effort of crafting their own attempts, and have been thrilled, after this investment of effort, to have found the bridge that elegantly takes them across to the next problem, the next conceptual area, the next vista for understanding.

> "By producing examples and by observing the properties of special mathematical objects, one could hope to obtain clues as to the behavior of general statements which have been tested on examples."
> – S.M. Ulam, Adventures of a Mathematician

Laying out mathematics in this way does double credit: i) the student builds incomparably better gut-level understanding of the material because they have themselves done the guided exploring; ii) the field of mathematics and the goodwill toward mathematicians have also benefited: the individual mathematicians associated with progress in each area are no longer obscure, but are part of the fabric of the student's experience of mathematics as a living history of intellectual ideas.

Preparing Future Graduates for the Mathematical Sciences

How can a general mathematics education entice students to the mathematical sciences? I am convinced it lies in large part in empowering students to attempt to solve real problems and experience first-hand the pleasure (and accompanying frustration) of how mathematics is explored.

As students become bolder through their guided explorations around an area, some will find that they have discovered for themselves, in a flush of excitement, their own way across. This is enormously empowering and provides a confidence that is difficult to imagine without having experienced it. It is precisely these

[4] In this age when a public display of excitement is anathema to teenagers, the signs of excitement are much more subdued. But they can still be seen. The eyes and behavior show immediately when a student is engaged and when he has switched off.

kinds of boosts that create in students the courage to hold high ambitions, and to consider mathematics and science as holding the possibility of a fulfilling future.

> The interplay between generality and individuality, deduction and construction, logic and imagination—this is the profound essence of live mathematics. Any one or another of these aspects of mathematics can be at the center of a given achievement. In a far reaching development all of them will be involved. Generally speaking, such a development will start from the "concrete" ground, then discard ballast by abstraction and rise to the lofty layers of thin air where navigation and observation are easy; after this flight comes the crucial test of landing and reaching specific goals in the newly surveyed low plains of individual "reality". In brief, the flight into abstract generality must start from and return again to the concrete and specific. – Richard Courant (mathematician)

This appreciation, pitched at the appropriate level for each audience, is what I believe should be the goal of teaching mathematics to general education students. These students may choose not to continue into technical mathematics coursework, but they will nonetheless have their life enriched and be better prepared to participate in the modern quantitive world if they have had a positive, profound experience with mathematics as an intellectual, humanistic subject.

This is what I hope every student experiences of mathematics.[5]

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[5] With regards to technical mathematics courses: there is certainly a valid place for teaching technical mathematics in a manner that efficiently covers the details of the theory and practice of a branch of professional mathematics. But I believe that even the details of such a course would be better internalized if its students understood the context of the technical mathematics being taught.[Tao] Thus, even for an audience of potential mathematicians, I contend that the four points outlined in this article are still valid, and would do well to be brought into a technical course, though certainly to a different degree than in a first encounter with a mathematical subject.

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