What is Mathematics?

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Can a single definition be found that captures the meaning of Mathematics across the millennia of its recorded history? What unites the practice of mathematics throughout its history and into the present time?

In this article, I will try for a short answer by proceeding iteratively. Convergence will arrive in two iterations.

What is Mathematics?

Though mathematical understanding is ancient, stretching back into pre-history, its concepts, organization, scope, outlook and practice have seen profound evolution over the millenia.

So what *is* this thing called mathematics? Can a single definition be found that captures the meaning of Mathematics across the millennia of its recorded history? What unites the practice of mathematics throughout its history and into the present time?

Getting at a satisfactory answer is slippery business. What adequately describes mathematics at various earlier periods of its history is typically inadequate for contemporary mathematics.[1] The same is true in reverse: an abstract, structure based discussion of mathematics that fits contemporary mathematics would exclude large periods in the early history of mathematics. [CC09].

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[1]Even a cursory look at the modern mathematical literature and its applications, or at the subject classification of the AMS makes it abundantly clear that mathematics covers an enormous breadth of material today and that mathematicians are investigating these areas in many different ways.

Criteria for a Good Definition of Mathematics

What should a good answer to the question "What is Mathematics?" look like? I believe it should hold up well against the following three criteria:

1. it should cover the practice of Mathematics through its history and into the present.

   This means it should apply, not only to the broad range of pure and applied mathematics today, but also to primitive, ancient, and classical mathematics — the mathematics of the Babylonians, Egyptians, Greeks, Indians, Chinese, Arabs, Central Asians, and Mercantile-Period Italians — as well as to the mathematics that accompanied the flowering of science and technology in the modern age.

2. it should not exclude periods or activities that do not meet current standards of rigor, or those from a particular historical time;

3. it should be in harmony with the manner by which mathematical knowledge develops, both in the past and presently, and with the perspectives of the various users of mathematics today: scientists, engineers, applied and pure mathematicians; and

In this article, I will try for a short answer by proceeding iteratively. Convergence will arrive in two iterations. (For answers that are essentially surveys of mathematics and its variations, directions, and forms, see [Ale56], [DH81], [CR41], and [Mac86].))

First Iteration

I'll start with a one-liner that has basic versions of the first four out of five ingredients:

> Mathematics is a subject concerned with number, shape, change, and relation...

Unpacking this:

- Number has to do with quantity, measurement, and scale;
- Shape is about configuration and arrangement;
- Change considers time and variation; and
- Relation has to do with association and comparison (similarity, difference, equality).

These first four areas, taken together, adequately cover proto-mathematics and ancient mathematics, including the mathematics of ancient Babylonia, Egypt, China, and India: numeration systems, integer arithmetic, non-symbolic solution of linear equations and select quadratic equations, mensuration, the properties of various two and three dimensional figures, the fundamentals of statics and mechanics, the keeping of time, division of property, taxation, and arithmetic of fractions.

Second Iteration

For the rest of mathematics, we'll need more. To cover classical mathematics through to contemporary mathematics requires a tweak to the basic ingredients, the addition of a fifth crucial ingredient, and some exposition:

> Mathematics is a subject concerned with four natural phenomena: quantity, space, transformation, and relation. Deepening understanding of the four natural phenomena leads to the development of a chain of evolving conceptual abstractions, to greater generalization, and to correspondingly broader areas of investigation.
>
> This has created a fifth, humanistic area of mathematical activity: the development of logically structured (axiomatic) mathematical systems that generalize and extend empirical concepts, introduce fundamentally new theoretical concepts, and examine the laws that govern their structure, properties and relationships holding between them.
>
> Progress in this last area has led to the development of a variety of mathematical systems. The breadth of modern mathematics is organized within these mathematical systems, and it is from within these systems that they find application in areas beyond the historical core.

Unpacking this:

The fifth, humanistic area of mathematics, structure, has to do with the development of mathematical systems to organize empirical or informal mathematical knowledge, axiomatic foundations for the various areas of mathematical activity, and establishing standards of logical inference and proof.

With its addition, we can cover classical mathematics, including the mathematics of Greece, the golden age of geometric science in Hellenistic North Africa, and the golden age of science in Arabia, Central Asia, and India: the development of axiomatic geometry and associated progress in plane geometry, solid geometry, and analysis, the development of triangle geometry (trigonometry), a systematic development of algebra, the algorithmization of arithmetic, and developments foreshadowing the calculus and symbolical mathematics.[2]

[2]The Greeks encountered the paradoxes of the real numbers, though not their resolution, and were already using the method of exhaustion and converging upper and lower bounds, a precursor to the methods of the integral calculus, apart from a rigorous passing to the limit.

The Arab and Central Asians and Indians advanced algebra to an abstract science, had resolved the solution of algebraic equations including most instances of the general cubic, had developed expansion by Taylor series and precursors of the calculus, had developed the

Once we come to pre-modern and modern mathematics, cross-pollination betweeen areas of mathematics and the earnest development of mathematical structures means that most areas are now associated with various combinations of the five areas:

- Quantity continues to hold matters of number, measurement, and scale. But it is also the subject of arithmetic, geometry, analysis, complex variables, combinatorics, probability, statistics, optimization — indeed almost all of mathematics involves quantity or one of its conceptual abstractions.

- Space continues to hold matters to do with shape, configuration, arrangement, symmetry, perspective. But it is also the subject of the continuum in one, finitely many, and infinite dimensions, the subject of geometry, continuous group theory, linear algebra, analysis, differential mathematics, topology, and set theory, among others.

- Change continues to be about about time, but also includes mathematical tools for representing and analyzing transformation: functions, analytic geometry, the calculus, differential mathematics, geometrical physics, matrix algebra, and analysis, among others.

- Relation continues its association with comparison, similarity and equality, but extends these more generally to equivalence, and the subjects of abstract algebra, set theory, and logic, among others.

- Structure permeates all of modern mathematics, organizing, arranging, rigorizing informal mathematical knowledge, providing axiomatic foundations for the various areas of mathematical activity, and acceptable standards of logical inference and proof. The development and refinement of mathematical structures is never done, and work continues in modern logic, proof theory, set theory, as well as in the intersections between mathematics and physics (gauge theories, string theories, quantum field theories), computer science (computability, recursion, mathematical linguistics), biology (chaos theory, self-organizing systems), and in many other areas.

Three Facets of Mathematics

A definitional answer, unfortunately, is just a start to explaining Mathematics.

To understand Mathematics in a way that is consistent with its history, evolution, and its many diverse applications, as well as with its contemporary, abstract, and highly specialized state, it is helpful to identify three co-existing facets of Mathematics:

trigonometric identities, applied algebra and trigonometry to astronomical problems, and had fully developed systems of computation for interest rates, taxation, and other numerical calculations using decimal digits including zero, and a place decimal system.

1. Mathematics as an empirical science,
2. Mathematics as a modeling art, and
3. Mathematics as an axiomatic arrangement of ideas, their relations, and the conceptual structures built around them.

These facets of Mathematics explain both the historical development, maturity, and modern separation between theoretical and applied considerations.

Mathematics as an Empirical Science

Mathematics originates out of science, i.e. out of human interest in the surrounding world, its careful observation, and the empirical verification of mathematical fact. In fact, the world and its patterns are consistently present in the inspiration of all mathematical studies. Even the abstract, abstruse, and seemingly detached topics of advanced higher mathematics are generalizations of patterns observed in the layers of less abstract mathematics, that are themselves an attempt to capture patterns observed in the real world itself.

I would venture, therefore, that the essence of a mathematical concept can always be related back to an original proto-concept that has its roots in empirical observations and the patterns arising out of these.

Mathematics as a Modeling Art

Mathematics as a modeling art involves an effort to develop, maintain, and perfect models of perceived or envisioned reality.

At the root of this facet of mathematics is the intimate relationship between the physical world and the world of mathematical ideas. Most of the major laws of mathematics are modeled on actual physical occurrences, suitably abstracted. Thus, I would argue that the origin of most of mathematics is a model of something real that has been experienced. Indeed, typical applied mathematics proceeds from a physical context to the context of a mathematical model, performs computations and analysis using mathematical reasoning within the domain of this model, and then finally brings the result back to the physical context for interpretation.

The success of Mathematics in keeping ever-improving mathematical models of many and various phenomena, and the fact that the methods behind these models are often applicable in widely different areas and contexts, often lead Mathematics to be viewed enthusiastically (though incorrectly) as the key to the knowledge of all things.[3]

[3] Successful generalization often leads to that heady feeling of lifting the veils from the face of Mysteries. Indeed, the Rhind Papyrus containing mathematical knowledge of ancient Egyptian begins with a description of what it discusses:

> "a thorough study of all things, insight into all that exists, knowledge of all obscure secrets." ([Bur], p.38.)

The mathematical knowledge contained in the Rhind Papyrus, the ability to compile, present,

Mathematics as an Axiomatic Arrangement of Knowledge

The exploration of the logical structure of mathematical knowledge is a relatively recent development, beginning with the ancient Greeks circa 800 BCE. Comparatively, this phenomenon has occupied less than 3 millenia, or less than 10% of the documented history of mathematical knowledge of humankind (30,000 years).

Rapid progress in understanding the logical structure of mathematics occurred since the 1800s CE,[4] and has led to the flowering of a wide variety of modern mathematical systems and theories whose areas of interests and domains of application go far beyond the historical core of mathematics.

Today, the vast scope of modern mathematical knowledge is organized within structured mathematical systems from within which it finds wide application.

Mathematical structures distill informal mathematical knowledge, identify the important concepts out of the body of informal mathematical knowledge, and provide streamlined logical models to underpin these areas.

Characteristics of Modern Mathematics

I have argued that Mathematics arises from Man's attempt to summarize the variety of empirical phenomena that he experiences, and that Mathematics advances through the expansion and generalization of these concepts, and the improvement of these models.

But what are the *characteristics* of mathematics, especially modern mathematics?

I'll consider five groups of characteristics:

1. Applicability and Effectiveness,
2. Abstraction and Generality,
3. Simplicity,
4. Logical Derivation, Axiomatic Arrangement,
5. Precision, Correctness, Evolution through Dialectic.

Though each of these characteristics presents unique pedagogical challenges and opportunities, here I'll focus on the characterisics themselves and leave the pedagogical discussion to [Ebr05].

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and solve a compendium of practical mathematical exercises using abstract techniques of multiplication and division, was certainly prized among the ancient Egyptians, and known to a limited few. But a study of *all* things? insight into *all* that exists? knowledge of *all* obscure secrets? Anyone who has tasted the heady feeling of succesful generalization and a survey of a branch of the science will perhaps understand the author's excess of enthusiasm for the mathematical knowledge he was about to share.

[4] The 200 years of mathematics since 1800 CE is less than 1% of the documented history of mathematics.

Wide Applicability and the Effectiveness of Mathematics

General applicability is a recurring characteristic of mathematics: mathematical truth turns out to be applicable in very distinct areas of application in phenomena from across the universe to across the street. Why is this? What is it about mathematics and the concepts that it captures that causes this?

Mathematics is widely useful because the five phenomena that it studies are ubiquitous in nature and in the natural instincts of man to seek explanation, to generalize, and to attempt to improve the organization of his knowledge. As Mathematics has progressively advanced and abstracted its natural concepts, it has increased the host of subjects to which these concepts can be fruitfully applied.

Abstraction and Generality

Abstraction is the generalization of myriad particularities. It is the identification of the essence of the subject, together with a systematic organization around this essence. By appropriate generalizations, the many and varied details are organized into a more manageable framework. Work within particular areas of detail then becomes the area of specialists.

Put another way, the drive to abstraction is the desire to unify diverse instances under a single conceptual framework. Beginning with the abstraction of the number concept from the specific things being counted, mathematical advancement has repeatedly been achieved through insightful abstraction. These abstractions have simplified its topics, made the otherwise often overwhelming number of details more easily accessible, established foundations for orderly organization, allowed easier penetration of the subject and the development of more powerful methods.

Simplicity (Search for a Single Exposition), Complexity (Dense Exposition)

For the outsider looking in, it is hard to believe that simplicity is a characteristic of mathematics. Yet, for the practitioner of mathematics, simplicity is a strong part of the culture. Simplicity in what respect? The mathematician desires the simplest possible *single* exposition. Through greater abstraction, a single exposition is possible at the price of additional terminology and machinery to allow all of the various particularities to be subsumed into the exposition at the higher level.

This is significant: although the mathematician may indeed have found his desired single exposition (for which reason he claims also that simplicity has been achieved), the reader often bears the burden of correctly and conscientiously exploring the quite significant terrain that lies beneath the abstract language of the higher-level exposition.

Thus, I believe it is the mathematician's desire for a single exposition that leads to the attendant complexity of mathematics, especially in contemporary mathematics.

Logical Derivation, Axiomatic Arrangement

The modern characteristics of logical derivability and axiomatic arrangement are inherited from the ancient Greek tradition of Thales and Pythagoras and are epitomized in the presentation of Geometry by Euclid (The Elements).

It has not always been this way. The earliest mathematics was firmly empirical, rooted in man's perception of number (quantity), space (configuration), time, and change (transformation). But by a gradual process of experience, abstraction, and generalization, concepts developed that finally separated mathematics from an empirical science to an abstract science, culminating in the axiomatic science that it is today.

It is this evolution from empirical science to axiomatic science that has established derivability as the basis for mathematics.

This does not mean that there is no connection with empirical reality. Quite the contrary. But it does mean that mathematics is, today, built upon abstract concepts whose relationship with real experiences is useful but not essential. These abstractions mean that mathematical fact is now established without reference to empirical reality. It may certainly be influenced by this reality, as it often is, but it is not considered mathematical fact until it is established according to the logical requirements of modern mathematics.

Why the contemporary bias for axiomatic Mathematics? Why is axiomatic mathematics so heavily favored by modern mathematics? For the same reason it was favored in the time of Euclid: in the presence of empirical difficulties, linguistic paradox, or conceptual subtlety, it is an anchor that clarifies more precisely the foundations and the manner of reasoning that underlie a mathematical subject area. Once the difficulties of establishing an axiomatic framework have been met, such a framework is favored because it helps ease the burden of many, complicated, inter-related results, justified in various ways, and inter-mixed with paradoxes, pitfalls, and impossible problems. It is favored when new results cannot be relied upon without complicated inquiries into the chains of reasoning that justify each one.

The value of axiomatic mathematics What the axiomatic approach offers is a way to bring order to a subject area, but one which requires deciding what is fundamental and what is not, what will be set up higher as a "first principle" and what will be derived from it. When it is done, however, it sets a body of knowledge into a form that can readily be presented and expanded. Appealing and effective axiom systems are then developed and refined. Their existence is a mark of the maturity of a mathematical subfield. Proof within the axiomatic framework becomes the hygiene that the community of working mathematicians adopts in order to make it easier to jointly share in the work of advancing the field.[5]

Axiomatic Mathematics as Boundaries in the Wilderness In all cases of real mathematical significance, the selection of axioms is a culminating result of intensive investigations into an entire mathematical area teeming with phenomena, and the gaining of a deep understanding that results finally in identifying a good way to separate the various phenomena that have been discovered. So, though the axioms may sound trivial, in reality, the key axioms delineate substantially different structures. In this sense, axioms are boundaries that separate structurally distinct areas from each other, and, together, from the rest of "wild" mathematics.

For example: the triangle inequality is a theorem of Euclidean geometry. But it is taken as an axiom for the study of metric spaces. By doing so, this one axiom forces much of the Euclidean isometric structure. As such, it becomes a code or litmus test for the "Euclidean-ness" of a space.

Thus, from this point of view, non-axiomatic mathematics is the mathematics of discovery. Axiomatic mathematics has been tamed and made easy to learn, present, and work within. One might regard it is a fenced off area within the otherwise unmarked wilderness of other mathematical and non-mathematical phenomena. One might think of the progress of axiomatic mathematics as paralleling the way in which mankind slowly but inexorably tamed the wilderness, chopping down the trees and pushing the truly wild animals further away, while domesticating and harnessing the desirable easier ones, and setting up buildings, and walkways, farmlands, granaries, and a functioning and productive economy.

The same holds for mathematical definitions: they are attempts to tame certain phenomena and identify them as the subjects for further domestication and as able to be safely put to use, shutting out the untamed disorder of the rest of phenomena, mathematical and non-mathematical.

The down-side of axiomatics One down-side of the axiomatic presentation of mathematics is that although deep understanding is typically hidden within the axioms, the definitions of the mathematical systems have been designed precisely to make the axioms seem trivial. Which means that it is all too easy to simply state them and move on to the "meat" of the matter.

But this would be a mistake. Time spent understanding why the axioms are there, seeing them as theorems in historically prior investigations, and understanding in what phenomena they arise and where they don't — time spent this way leads to a much deeper understanding of the significance of taking the

[5] What is axiomatized now? Logic (Standard logics – propositional, predicate; non-standard logics – modal, etc.). Set theory (Standard – Zermelo-Fraenkel; enhanced – Zermelo-Fraenkel-Axiom of Choice; non-standard). Algebraic systems (Groups, rings, fields, and all manner of systems between these, including arithmetic, and the integers). Analysis (Dedekind cuts of the rational or equivalent Cauchy sequences of rationals). Topology (all manner of spaces). Modern Geometry is now an application of all of these axiomatized mathematical fields to the studies of space.

axioms on in the first place and understanding the boundaries of the subject that the axioms establish.

Axiomatic mathematics and density of presentation For those who are interested in learning mathematics efficiently, the axiomatic presentation is most definitely the most efficient both in presentation and in "coverage density" (you get the most amount of reach and the greatest applicability in the fewest steps).

But along with "coverage density" comes conceptual density. The abstract language of axiomatic mathematics can subsume vastly different specific examples within a single abstract statement, examples which may spread across a host of historic sub-disciplines and mathematical objects of interest. You may follow the proof, and be able to turn out your own (within the framework of the axiomatics), but do you really understand the results deeply? Have you actually rubbed shoulders with the individual mathematical animals? Would you be able to recognize the right way to subdue a specific animal if it came across your path in unfamiliar circumstances (i.e. not presented in the efficient abstract language)?

Precision, Correctness, Evolution Through Dialectic

The Language of Mathematics. Over the course of the past three thousand years, mankind has developed sophisticated spoken and written natural languages that are highly effective for expressing a variety of moods, motives, and meanings. The language in which Mathematics is done has developed no less, and, when mastered, provides a highly efficient and powerful tool for mathematical expression, exploration, reconstruction after exploration, and communication. Its power (when used well) comes from simultaneously being precise (unambiguous) and yet concise (no superfluities, nothing unnecessary). But the language of mathematics is no exception to being used poorly. Just as any language, it can be used well or poorly.

Once correctness in mathematics is separated from empirical evidence and moved into a model-based or axiomatic framework, the touchstone for correctness becomes other, carefully selected, statements that capture the essential elements of the underlying reality: definitions, axioms, previously established theorems. The language of mathematics, and logical reasoning using that language, form the everyday working experience of mathematics.

Symbolical mathematics. In earlier times, mathematics was in fact, fully verbal. Now, after the dramatic advances in symbolism that occurred in the Mercantile period (1500s), mathematics can be practised in an apparent symbolic shorthand, without really the need for very many words. This, however, is only a shorthand. The symbols themselves require very careful and precise definition and characterization in order for them to be used, computed with, and allow the results to be correct.

The modern language of *working* mathematics, as opposed to expository or pedagogical mathematics, is symbolic, and is built squarely upon the propositional logic, the first order predicate logic, and the language of sets and functions.[6] The symbolical mode is one which should be learned by the student and used by the practitioner of mathematics. It is the clearest, most unambiguous, and so most precise and therefore demanding language. But, one might say it is a "write only" language: you don't want to read it. So, once one has written out ones ideas carefully this way, then one typically switches to one of the other two styles: direct or expository, these being the usual methods of communicating with others.[7]

Evolution Through Dialectic Mathematical definitions, mathematical notions of correctness, the search for First Principles (Foundations) in Mathematics and the elaboration of areas within Mathematics have all proceeded in a dialectic fashion, alternating between periods of philosophical/foundational contentment coupled with active productive work on the one hand, and the discovery of paradoxes coupled with periods of critical review, reform, and revision on the other. This dialectical process through its history has progressively raised the level of rigor of the Mathematics of each era.[8]

The level of precision in mathematics increased dramatically during the time of Cauchy, as those demanding rigor dominated mathematics. There were simply too many monsters, too many pitfalls and paradoxes from the monsters of functions in the function theory to the paradoxes and strangenesses in the Fourier analysis and infinite series, to the paradoxes of set theory and modern logic. The way out was through subtle concepts, subtle distinctions, requiring careful delineations, all of which required precision.

A Culture of Precision Mathematical culture is that what you say should be correct. What you say should have a definition. You should know the definition and limits of what you are saying, stating, or claiming. The distinction is between mathematics being developed informally and mathematics being done more formally, with necessary and sufficient conditions stated up front and restricting the discussion to a particular class of objects.

Thus, I would argue that the modern mathematical culture of precision arises because:

1. mathematics has developed a precise, highly symbolic language,
2. mathematical concepts have developed in a dialectic manner that allows for the adaptation, adjustment and cumulative refinement of concepts based on experiences, and
3. mathematical reasoning is expected to be correct.

[6] I discuss the details of mathematical language, set theory, and logic, in a separate article.

[7] See [Lam95], and [Dij89b] for discussions by mathematicians on the symbolic mode and its advantages.

[8] See [Lak76]

Further Reading

The following books and papers are recommended for additional reading on the topics discussed in this article.

- An expository survey of elementary mathematics provides an excellent example of the richness and creativity of mathematical ideas, as well as a condensed glimpse into the evolution of mathematical ideas. See [Ebr06] and [Ale56].
- An expository look at the questions of axiomatic foundations of mathematics is contained in the short paper [Fef99].
- Lakatos develops a particularly vivid presentation of dialectic in mathematics in his mathematical-literary play *Proofs and Refutations*, [Lak76].
- Lamport and Djikstra inquire into the place and method of proof in mathematics in [Lam95] and [Dij89b].

Other recommended references are:
[Bou], [Bri57], [Bri59], [Bri62], [Bul94], [Bur], [CR41], [DH81], [Dij89a], [Dij], [Dij98], [Ebr04], [Ebr08], [Fef], [Fef98], [Fef92b], [Fef99], [Fef92a], [Gal94], [GKP], [Guged], [Gul97], [Hal87], [GKHK75], [Tuc04], [Dor], [FD07], [Kle86], [Kli], [Pan], [Ped89], [Rot97], [Rus], [Wal06], [Wil82], [Zad75].

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